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Normal forms of para-CR hypersurfaces ☆

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A B S T R A C T

We consider hypersurfaces of finite type in a direct product space $\mathbb{R}^2 \times \mathbb{R}^2$, which are analogues to real hypersurfaces of finite type in $\mathbb{C}^2$. We shall consider separately the cases where such hypersurfaces are regular and singular, in a sense that corresponds to Levi degeneracy in hypersurfaces in $\mathbb{C}^2$. For the regular case, we study formal normal forms and prove convergence by following Chern and Moser. The normal form of such an hypersurface, considered as the solution manifold of a 2nd order ODE, gives rise to a normal form of the corresponding 2nd order ODE. For the degenerate case, we study normal forms for weighted ℓ-jets. Furthermore, we study the automorphisms of finite type hypersurfaces.

1. Introduction

It is well known that the $2n+1$-dimensional Heisenberg group, viewed with its CR structure, is the Šilov boundary of the Siegel domain in $\mathbb{C}^{n+1}$, namely the quadric $\Im w = |z|^2$, $z \in \mathbb{C}^n$. It is also well known that the infinitesimal automorphisms of the Heisenberg group with its CR structure form the Lie algebra $\mathfrak{su}(1, n+1)$. On the one hand, the Heisenberg algebra is the nilpotent component of the Iwasawa decomposition of $\mathfrak{su}(1, n+1)$. On the other hand, when $n = 1$, the three dimensional Heisenberg algebra is also the nilpotent component of the Iwasawa decomposition of $\mathfrak{sl}(3, \mathbb{R})$. This fact has a nice geometric interpretation: the three dimensional Heisenberg group with its multicontact structure is diffeomorphic to the hypersurface $y = a + bx$ of $\mathbb{R}^2_{xy} \times \mathbb{R}^2_{ab}$ endowed with its para-CR structure [15]. For more on multicontact structures see [1,3–5,7,11,14], see also [17,18]. The infinitesimal automorphisms of this structure on $y = a + bx$ are given

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by $\mathfrak{sl}(3,\mathbb{R})$. As one does for CR manifolds, it is then natural to study general hypersurfaces embedded in the product space $\mathbb{R}^2_{xy} \times \mathbb{R}^2_{ab}$ and consider their infinitesimal automorphisms and normal forms.

In this order of ideas, we consider hypersurfaces of the form $S : y = F(a,b,x)$, with $\frac{\partial F}{\partial a}(0,0,0) \neq 0$, and define a para-CR structure on S as the structure on TS induced by the embedding, namely the two direction fields $TS \cap T\mathbb{R}^2_{xy}$ and $TS \cap T\mathbb{R}^2_{ab}$. The para-CR hypersurface is regular if the commutator of the two direction fields generates the missing direction in TS at each point. This is analogous to Levi non-degeneracy of a CR manifold.

In this paper we study normal forms of regular para-CR hypersurfaces, in the spirit of Chern and Moser [2]. This leads to a normal form of second order ODE. We have learnt recently that I. Kossovskiy and D. Zaitsev [12] obtained independently a result on the classification of second order ODE's. Furthermore, we also study the formal normal forms for non-regular hypersurfaces of finite type, following ideas in [9]. Last but not least, we compute the infinitesimal automorphisms of para-CR hypersurfaces of finite type, completing the study that was started in [15].

The paper is organised as follows. After establishing the notation, we study in Section 3 the normal forms for regular hypersurfaces. First, we prove in Theorem 1 that the normal form can be achieved by weighted jets of the function defining the surface, and then we show that the normalisation is convergent in Theorem 2. In Section 3.2, we interpret a surface $S : y = F(a,b,x)$ as the space of solutions of a second order differential equation in one variable $y''(x) = B(x,y,y')$, with a,b being parameters indicating the initial conditions for y and y'. In Proposition 1, we show how the normal form for S reflects into a normal form of the function B. In Section 4, we consider the case of hypersurfaces S that are not regular. We define a normal form for the jets of the defining equation and prove in Theorem 3 that every such jet can be put into normal form. Finally, in Section 5, we apply the normal form to the study of the automorphisms of a hypersurface of finite type. The automorphisms for the model cases, in which F is a polynomial, were studied in [15]. Here we study the non-model case. In this setting, we show in Theorem 4 that there are no nontrivial isotropic automorphisms unless the Taylor expansion of the defining function F in normal form consists of monomials of the type $(b^m x^n)^r$, where m,n are fixed integers. In the latter case, we show that there is exactly a 1-parameter group of automorphisms.

2. Notation

We consider smooth hypersurfaces in $\mathbb{R}^4$ of the form $S : y = F(a,b,x)$, for which we assume $\frac{\partial F}{\partial a}(0,0,0) \neq 0$. This embedding distinguishes two direction fields $TS \cap T\mathbb{R}^2_{xy}$ and $TS \cap T\mathbb{R}^2_{ab}$ on S, which in local coordinates x,a,b take the form

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial F}{\partial a}\frac{\partial}{\partial b} - \frac{\partial F}{\partial b}\frac{\partial}{\partial a}.$$

We call a hypersurfaces in $\mathbb{R}^4$ with two distinguished vector fields X,Y a *para-CR hypersurface*. Sometimes it will be convenient to denote the partial derivatives of a function with the subscripts, in which case we shall write F_b, F_x, F_{bx}, and so on.

We say that S is of *finite type* $k \geq 2$ if k is the smallest integer such that $\frac{\partial^k F}{\partial^m b \partial^n x}(0,0,0) \neq 0$ for some $m,n > 0$ such that $m+n=k$, provided such a k exists. We say that S is *regular* if $k=2$, otherwise we call S *singular*. Given S of finite type k, let $F_\ell(a,b,x)$ be the polynomial such that

$$F(t^k a, tb, tx) = F_\ell(t^k a, tb, tx) + o(t^\ell),$$

as $t \to 0$, and call F_ℓ the *weighted ℓ-jet* of F. Denote by $R_\ell = F - F_\ell$ the remainder of weight greater than ℓ. We say that a polynomial $P(a,b,x)$ has *weight* ℓ if $P(t^k a, tb, tx) = t^\ell P(a,b,x)$. After a polynomial coordinate change, we may write