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Manifolds with many hyperbolic planes

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ABSTRACT

We construct examples of complete Riemannian manifolds having the property that every geodesic lies in a totally geodesic hyperbolic plane. Despite the abundance of totally geodesic hyperbolic planes, these examples are not locally homogeneous.

1. Introduction

Hyperbolic n-space is the space $\mathbb{H}^n = \{(x_i) \in \mathbb{R}^n \,|\, x_n > 0\}$ with the metric $h_n = x_n^{-2}(dx_1^2 + \cdots + dx_n^2)$. Each tangent 2-plane to $(\mathbb{H}^n, h_n)$ exponentiates to a totally-geodesic copy of $(\mathbb{H}^2, h_2)$. Similarly, hyperbolic surfaces abound in the negatively curved locally symmetric manifolds. More precisely, these spaces satisfy:

(P): *Each tangent vector to M is contained in a tangent 2-plane σ that exponentiates to an immersed totally-geodesic hyperbolic surface.*

We construct examples of manifolds with (P) that are *not* locally homogeneous. For $n \geqslant 2$, let $M^n = \mathbb{R} \times \mathbb{H}^{n-1} = \{(t, (x_i)) \,|\, t \in \mathbb{R},\, (x_i) \in \mathbb{H}^{n-1}\}$.

Main Construction. *For each $(a, b) \in \mathbb{R} \times [-1, 1]$, M^n admits a complete Riemannian metric $h_{n,a,b}$ satisfying:*

(1) *For fixed $n \geqslant 2$, the metrics $h_{n,a,b}$ depend smoothly on (a, b).*
(2) *There is a monomorphism* $\mathrm{Isom}(\mathbb{H}^{n-1}, h_{n-1}) \to \mathrm{Isom}(M^n, h_{n,a,b})$.

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(3) *Each tangent* 2*-plane to* M^n *containing the tangent vector* $\frac{\partial}{\partial t}$ *exponentiates to a totally-geodesic surface isometric to the hyperbolic plane.*
(4) *The sectional curvature of a* 2*-plane* σ *at* $p = (t, (x_i))$ *making angle* $\theta \in [0, \frac{\pi}{2}]$ *with* $\frac{\partial}{\partial t}$ *is given by*

$$\sec(\sigma) = -1 + \frac{4(1 - b^2 - e^{2a})}{((1+b)e^t + (1-b)e^{-t})^2} \sin^2(\theta).$$

(5) *For* $n \geqslant 3$, $(M^n, h_{n,a,b})$ *Riemannian covers a finite volume manifold if and only if* $1 = b^2 + e^{2a}$, *or equivalently, if and only if* $(M^n, h_{n,a,b})$ *is isometric to* $(\mathbb{H}^n, h_n)$.

Remark 1.1. Property (3) implies (P).

Remark 1.2. For $n \geqslant 3$, the Riemannian metric $h_{n,a,0}$ has extremal sectional curvatures -1 and $-e^{2a}$ by (4); consequently, manifolds satisfying (P) can have curvatures pinched arbitrarily close to -1 from above or below without being isometric to hyperbolic space.

The Main Construction is motivated by the study of *rank rigidity* [1,2,4–9,11,12], and in particular in manifolds of *positive hyperbolic rank* [5–8]. A complete Riemannian manifold M has *positive hyperbolic rank* if along each complete geodesic $\gamma : \mathbb{R} \to M$, there exists a parallel vector field $V(t)$ such that $\sec(\dot{\gamma}, V)(t) \equiv -1$. This hypothesis on geodesics is an infinitesimal analogue of (P).

The negatively curved locally symmetric spaces, normalized so that -1 is an extremal value of the sectional curvatures (i.e. an achieved lower or upper bound), have positive hyperbolic rank. While there are additional examples of manifolds with positive hyperbolic rank (such as in the Main Construction), hyperbolic rank-rigidity results assert that under additional mild hypotheses, the negatively curved locally symmetric spaces constitute all examples.

The known examples of complete and *finite volume* manifolds of higher hyperbolic rank are all locally symmetric. There are no additional complete and finite volume examples amongst manifolds having $\sec \leqslant -1$ [7]. The same is known to be true amongst the complete and finite volume non-positively curved Euclidean rank one manifolds with suitably pinched curvatures [6]. Finally, complete and finite volume three-dimensional manifolds of higher hyperbolic rank are real hyperbolic without any a priori sectional curvature bounds [8].

In contrast, it is known that complete *infinite volume* manifolds of higher hyperbolic rank need not be locally symmetric. The negatively curved symmetric spaces are characterized amongst the homogeneous manifolds with positive hyperbolic rank and $\sec \leqslant -1$ in [5]. Therein, a non-symmetric example is constructed. To our knowledge, the examples provided by the Main Construction are the first that are *not* locally homogeneous.

2. Warping hyperbolic 1-space over the Euclidean line

Let t denote the Euclidean coordinate on $\mathbb{R}$ and $r > 0$ the Euclidean coordinate on $\mathbb{H}^1$.

Consider the foliation of $\mathbb{H}^2$ by the family of (Euclidean) upper-half semicircles with common center the origin in $\mathbb{R}^2$. Each such semicircle, parameterized appropriately, is an h_2-geodesic: For each $r > 0$, the map

$$F_r : \mathbb{R} \to \mathbb{H}^2$$

defined by $F_r(t) = (r \tanh(t), r \operatorname{sech}(t))$ parameterizes the semicircle through $(0, r)$ in the clockwise fashion as a unit-speed h_2-geodesic with the initial point $(0, r)$.

Define a diffeomorphism

$$F : \mathbb{R} \times \mathbb{H}^1 \to \mathbb{H}^2$$