Differential Geometry and its Applications 52 (2017) 158–166

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Manifolds tightly covered by two metric balls

Jianming Wan [1]

School of Mathematics, Northwest University, Xi'an 710127, China

ARTICLE INFO

Article history:
Received 25 January 2017
Available online xxxx
Communicated by F. Fang

ABSTRACT

In this note we provide natural optimal geometric conditions for a Riemannian manifold suitably covered by two open metric balls to be homeomorphic to a sphere. This can be viewed as a geometric analogue of Brown's theorem in topology stating that a closed manifold covered by two topological balls is a sphere.

As the simplest closed manifold, the sphere enjoys a unique and basic role topologically as well as metrically. Geometrically, the unit sphere is uniquely determined as an "optimal object" in a variety of ways often referred to as *Sphere Theorems*. Examples of such recognition results include the classical Rauch–Berger–Klingenberg 1/4-pinching theorem (for diffeomorphism see Brendle–Schoen [5]), the diameter sphere theorem [15], Micallef–Moore's positive isotropic curvature sphere theorem [19], and Perelman's almost maximal volume sphere theorem [22] (for diffeomorphism see Colding and Cheeger [10,9]). Topologically, Brown's Theorem [4] recognizes the sphere as the only closed manifold covered by two open Euclidean balls.

As a metric contrast to Brown's Theorem, we point out, that any closed (smooth) manifold, M admits a Riemannian metric so that it is covered by two (proper) open metric balls, even *tightly covered* in the following sense:

For any $\epsilon > 0$ and fixed $r > 0$, there is a Riemannian metric on M so that

$$M = B(p, r + \epsilon) \cup B(q, r + \epsilon), \text{ with } \rho(p, q) = 2r$$

where $B(p, r)$ denotes the open r-ball centered at p and $\rho(p, q)$ is the distance between p and q. For example, it can be arranged that the complement of an arbitrarily small metric ball in M is a disc with constant curvature 1.

However, if for a fixed Riemannian metric M is ϵ-tightly covered for every $\epsilon > 0$, then of course

$$M = D(p, r) \cup D(q, r), \text{ with } \rho(p, q) = 2r,$$

E-mail address: wanj_m@aliyun.com.

[1] The author was supported in part by National Natural Science Foundation of China No. 11601421 and No. 11601422.

http://dx.doi.org/10.1016/j.difgeo.2017.04.003
0926-2245/

where $D(p,r)$ denotes the closed r ball with center p. In this case all geodesics emanating from p of length $2r$ are minimal and terminate at q (see Lemma 1.3). In particular, such an M is a topological sphere.

Our goal is to seek natural geometric conditions under which a Riemannian manifold tightly covered by two open metric balls, in the sense above, is a topological sphere. Our results hinge on the observation that for certain classes of metric spaces being tightly covered by two proper open metric balls is equivalent to having small excess in the sense of [13]. Here excess$M < \delta$ means there is a pair of points $p, q \in M$ such that for any $x \in M$,

$$\rho(p,x) + \rho(x,q) - \rho(p,q) < \delta.$$

In this case, clearly M is $\epsilon = \frac{\delta}{2}$ tightly covered in the above sense.

Indeed, we have (see section 1)

Theorem A. *Let $\mathcal{M}$ be a Gromov–Hausdorff precompact class of closed Riemannian manifolds for which any $X \in \bar{\mathcal{M}}$ is a non-branching geodesic metric space. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that excess$M < \epsilon$ if M is δ-tightly covered by two open balls, and vice versa.*

Recall, that from the Bishop–Gromov relative volume comparison theorem it follows that the class of all closed n-manifolds M with Ricci curvature, $\operatorname{ric} M \geq (n-1)k$ and diameter $\operatorname{diam} M \leq D$ is Gromov–Hausdorff precompact. The subclasses where the sectional curvature $\sec M \geq k$, or the injectivity radius $\operatorname{inj} M \geq i$ are examples of $\mathcal{M}$ as above. In the first case because any limit object is an Alexandrov space, and in the second case the non-branching property was proved by Taylor in [23].

Appealing to the main theorems of [21] and [13] this yields the following immediate corollaries

Theorem B. *For any real $k, r > 0, i > 0$ and integer $n \geq 2$ there is an $\epsilon_0 = \epsilon_0(k,r,i,n)$ such the following holds: Any closed Riemannian n-manifold M with $\operatorname{ric} M \geq (n-1)k$, $\operatorname{inj} M \geq i$ and*

$$M^n = B_p(r+\epsilon) \cup B_q(r+\epsilon), \quad \rho(p,q) = 2r$$

is homeomorphic to $\mathbb{S}^n$ if $\epsilon < \epsilon_0$.

If the condition $\operatorname{inj} M \geq i$ is relaxed to $\operatorname{vol} M \geq v$, the conclusion fails as, e.g., the examples due to Anderson [1] shows. However, if at the same time $\operatorname{ric} M \geq (n-1)k$ is strengthened to $\sec M \geq k$ we have:

Theorem C. *For any real $k, r > 0, v > 0$ and integer $n \geq 2$ there is an $\epsilon_1 = \epsilon_1(k,r,v,n)$ such the following holds: Any closed Riemannian n-manifold M with $\sec M \geq k$, $\operatorname{vol} M \geq v$ and*

$$M^n = B_p(r+\epsilon) \cup B_q(r+\epsilon), \quad \rho(p,q) = 2r$$

is homeomorphic to $\mathbb{S}^n$ if $\epsilon < \epsilon_1$.

In these statements we have no explicit estimate for ϵ_i. Likewise, we do not prove that the open metric balls $B(p, r+\epsilon)$ and $B(q, r+\epsilon)$ in M are homeomorphic to the Euclidean n-ball. Although, Theorem A implies that Theorems B and C are equivalent to the main results in [21] and [13] we present alternate short proofs.

In contrast, if $2r = d = \operatorname{diam} M$ and $\operatorname{vol} M \geq v$ is strengthened to $\operatorname{inj} M \geq i$, we have a constructive proof that being ϵ-tightly covered implies small excess. Thus by critical point theory lemma 3 of [13] we have