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On a projective class of Finsler metrics with orthogonal invariance

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ABSTRACT

In this paper, we study a class of Finsler metrics, called generalized Douglas–Weyl (GDW) metrics, which includes Douglas metrics and Weyl metrics. We find a sufficient and necessary condition for an orthogonally invariant Finsler metric to be a GDW-metric. As its application, we show that a certain class of Finsler metrics with orthogonal invariance are Douglas metrics if they are GDW-metrics, generalized a theorem previously only known in the case of Weyl metrics.

1. Introduction

The Douglas curvature is one of the important quantities in projective Finsler geometry. There are two class of Finsler metrics with important Douglas curvature properties. The first one is Douglas metrics [5]. The second one is generalized Douglas–Weyl metrics [7,10].

A Finsler metric is called a *Douglas metric* if it has vanishing Douglas curvature. A Finsler metric is said to be of *generalized Douglas–Weyl type* (GDW type for short) if the rate of change of the Douglas curvature along a geodesic is tangent to the geodesic. Note that every Douglas metric must be of GDW type. Why do we call the second class the generalized Douglas–Weyl metrics? According to Sakaguchi's surprising result, all Weyl metrics (metrics of vanishing Weyl curvature) must be of GDW type [8]. It is well-known that a Finsler metric is a Weyl metric if and only if it is of scalar flag curvature, namely, the flag curvature $K(P, y) = K(x, y)$ is independent of the section P containing y.

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In 2007, Najafi–Shen–Tayebi found an equation that characterize GDW-metrics of Randers type [6]. For example, the following Randers metric on $\mathbb{B}^n(\nu)$

$$F = \sqrt{f(|x|)|y|^2 + \tau^2 f^2(|x|)\langle x, y\rangle^2} + \tau f(|x|)\langle x, y\rangle \tag{1.1}$$

is a generalized Douglas–Weyl metric, where f is any differentiable function and κ is a constant.

Finsler metrics in (1.1) satisfy

$$F(Ax,\, Ay) = F(x,\, y)$$

for all $A \in O(n)$, equivalently, the orthogonal group $O(n)$ acts as isometries of F. Such metrics are said to be *orthogonally invariant* (spherically symmetric in an alternative terminology in [2, 5]).

Any orthogonally invariant Finsler metric $F = F(x,\, y)$ can be expressed by $F(x,\, y) = |y|\phi\left(|x|,\, \frac{\langle x,\, y\rangle}{|y|}\right)$ [3]. Hence all orthogonally invariant Finsler metrics are general $(\alpha,\, \beta)$-metrics [11]. They include many Finsler metrics with nice curvature properties [2, 3, 4, 5]. A lot of non-trivial spherically symmetric metrics of scalar flag curvature had been constructed explicitly in [2].

In this paper, we find a sufficient and necessary condition for a spherically symmetric Finsler metric F to be a GDW-metric. Precisely, we show the following:

Theorem 1.1. *Let $F = |y|\phi\left(|x|,\, \frac{\langle x,\, y\rangle}{|y|}\right)$ be an orthogonally invariant Finsler metric on $\mathbb{B}^n(\nu) \subset \mathbb{R}^n$. Then F is a GDW-metric if and only if*

$$6Q(Q_s - sQ_{ss}) + \frac{1}{r}(Q_{rs} - sQ_{rss}) - \left[1 - 2Q(r^2 - s^2)\right] Q_{sss} = 0$$

where Q is given in the first equation of (2.2).

Note that spherically symmetric Finsler metric $F = |y|\phi\left(|x|,\, \frac{\langle x,\, y\rangle}{|y|}\right)$ is of scalar curvature if and only if $R_2 = 0$ [2,4]. On the other hand, we have

$$\frac{\partial R_2}{\partial s} = 6Q(Q_s - sQ_{ss}) + \frac{1}{r}(Q_{rs} - sQ_{rss}) - \left[1 - 2Q(r^2 - s^2)\right] Q_{sss}.$$

This verifies Sakaguchi's theorem for Finsler metrics with orthogonal invariance [8].

Very recently, H. Zhu has discussed a certain class of orthogonally invariant Finsler metrics. She showed that these metrics are Douglas metrics if they are of scalar flag curvature [12].

As an application of Theorem 1.1, we show Zhu's result holds for GDW-metrics with orthogonal invariance.

Theorem 1.2. *Let $F = |y|\phi\,(r,\, s)$ be an orthogonally invariant Finsler metric on $\mathbb{B}^n(\nu)$, where $r = |x|$ and $s = \frac{\langle x,\, y\rangle}{|y|}$. Assume that $Q = Q(r,\, s)$ is a polynomial function of k degree with respect to s defined by*

$$Q(r,\, s) = f_0(r) + f_1(r)s + \cdots + f_k(r)s^k.$$

Then F is a GDW-metric if and only if it is a Douglas metric.

According to T. Sakaguchi's result, all Weyl metrics are GDW-metrics [8]. Here we weaken Zhu's condition.

Finally, we should point out that Mo–Solorzano–Tenenblat obtain all the Douglas metrics with orthogonal invariance [6].