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Sectional curvatures of ruled real hypersurfaces in a complex hyperbolic space

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ABSTRACT

A ruled real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)\,(n \geq 2)$ of constant holomorphic sectional curvature $c(\neq 0)$ is, in a word, a real hypersurface having a foliation by totally geodesic complex hyperplanes $\widetilde{M}_{n-1}(c)$. In this paper, we investigate the sectional curvatures K of ruled real hypersurfaces in a complex hyperbolic space and show that such hypersurfaces are classified into two types with regard to the range of K.

1. Introduction

A complex n-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature $c(\neq 0)$ is called a nonflat complex space form, which is denoted by $\widetilde{M}_n(c)$. Such a space is holomorphically isometric to either an n-dimensional complex projective space $\mathbb{C}P^n(c)$ or an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ according as $c > 0$ or $c < 0$.

The study of real hypersurfaces isometrically immersed into $\widetilde{M}_n(c)$ is one of the most interesting fields in Riemannian submanifold theory. Among typical examples of real hypersurfaces in $\widetilde{M}_n(c)$, we have the class of ruled real hypersurfaces. A real hypersurface M^{2n-1} in $\widetilde{M}_n(c)\,(n \geq 2)$ is said to be *ruled* if the holomorphic distribution $T^0M := \{X \in TM \,|\, \eta(X) = 0\}$ is integrable and each of its leaves is locally congruent to a totally geodesic complex hypersurface $\widetilde{M}_{n-1}(c)$ of the ambient space $\widetilde{M}_n(c)$, where η is the

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contact form on M. Every ruled real hypersurface in $\widetilde{M}_n(c)$ can be constructed in the following manner. We take an arbitrary regular real curve $\gamma : I \to \widetilde{M}_n(c)$ parametrized by its arclength s defined on some open interval $I(\subset \mathbb{R})$. At each point $\gamma(s)\,(s \in I)$ we attach a complex hyperplane $M_s \cong \widetilde{M}_{n-1}(c)$ in such a way that the hyperplane M_s is orthogonal to the real plane spanned by $\dot\gamma(s)$ and $J\dot\gamma(s)$, where J denotes the Kähler structure of the ambient space. Then, we obtain a ruled real hypersurface $M = \bigcup_{s\in I} M_s$ in $\widetilde{M}_n(c)$. We call this M a ruled real hypersurface *associated with* γ. Since it may in general have singularities, we must omit such points.

Every ruled real hypersurface M must *not* be Hopf, that is, the characteristic vector field of M is not principal on its open dense subset (see Section 2). Moreover, this class of real hypersurfaces in a complex hyperbolic space $\mathbb{C}H^n(c)$ contains nice three examples, which are the minimal homogeneous one, the complete non-homogeneous minimal one and the non-complete non-homogeneous minimal one (for details, see [2]).

Needless to say that the sectional curvature is one of the most important geometric invariants in Riemannian geometry. It is well-known that every ruled surface in 3-dimensional Euclidean space $\mathbb{R}^3$ has nonpositive Gaussian curvature. So, we have naturally an interest in the sectional curvature of every ruled real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$. In [9], the authors showed that the sectional curvature K of every ruled real hypersurface in $\mathbb{C}P^n(c)$ satisfies a sharp inequality $-\infty < K \leq c$. In this paper, we pay attention to ruled real hypersurfaces in $\mathbb{C}H^n(c)$ associated with smooth regular curves and investigate the sectional curvatures K of them. We shall show that such real hypersurfaces M are classified into two types with regard to the range of K, that is, M satisfies sharp inequalities either $-\infty < K \leq c/4$ or $c \leq K \leq c/4$ (Theorem in Section 4).

2. Preliminaries

First of all, we set up some notations. Let M be a real hypersurface of an $n\,(\geq 2)$-dimensional nonflat complex space form $\widetilde{M}_n(c)$ endowed with Riemannian metric g and the canonical Kähler structure J through an isometric immersion. We denote by $\mathcal{N}$ a unit normal local vector field on M and by A the shape operator of M in $\widetilde{M}_n(c)$. Then the Riemannian connections $\widetilde\nabla$ of $\widetilde{M}_n(c)$ and ∇ of M are related by Gauss and Weingarten formulas

$$\begin{cases} \widetilde\nabla_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \\ \widetilde\nabla_X \mathcal{N} = -AX \end{cases} \tag{2.1}$$

for vector fields X and Y tangent to M. On the real hypersurface M, an *almost contact metric structure* (ϕ, ξ, η, g) associated with $\mathcal{N}$ is naturally induced as

$$\xi = -J\mathcal{N}, \quad \eta(X) = g(X, \xi) \quad \text{and} \quad \phi X = JX - \eta(X)\mathcal{N}$$

for each tangent vector $X \in TM$. We call ξ, ϕ and η the *characteristic vector field*, the *characteristic tensor field* and the *contact form* on M, respectively. It is well-known that they satisfy

$$\begin{cases} \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \\ \phi^2 = -id + \eta \otimes \xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \end{cases} \tag{2.2}$$

and

$$\nabla_X \xi = \phi AX, \tag{2.3}$$

for all vectors $X, Y \in TM$.