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The dual foliation of some singular Riemannian foliations

Yi Shi

School of Mathematical Sciences, Shanxi University, Taiyuan 030006, China

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ABSTRACT

In this paper, we use the methods of subriemannian geometry to study the dual foliation of the singular Riemannian foliation induced by isometric Lie group actions on a complete Riemannian manifold M. We show that under some conditions, the dual foliation has only one leaf.

1. Introduction

We recall some basic notions about singular Riemannian foliations, for further details we refer the readers to [1,8,13,16]. A *singular Riemannian foliation* $\mathcal{F}$ on a Riemannian manifold M is a decomposition of M into smooth injectively immersed submanifolds $L(p)$, called leaves, such that it is a singular foliation and any geodesic starting orthogonally to a leaf remains orthogonal to all leaves it intersects. Such a geodesic is called a *horizontal geodesic*. For all $p \in M$, we denote by H_p the orthogonal complement to the tangent space $T_p(L(p))$, and call it the horizontal space at p. If all the leaves have the same dimension, then $\mathcal{F}$ is called a *regular Riemannian foliation*.

A curve is called *horizontal* if it meets the leaves of $\mathcal{F}$ perpendicularly. In [16], Wilking associates to a given singular Riemannian foliation $\mathcal{F}$ the so-called *dual foliation* $\mathcal{F}^{\#}$. The *dual leaf* through a point $p \in M$ is defined as all points $q \in M$ such that there is a piecewise smooth, horizontal curve from p to q. We denote by $L_p^{\#}$ the dual leaf through p. Wilking proved that when M has positive curvature, the horizontal connectivity holds on M, i.e. the dual foliation has only one leaf. Wilking also used the theory of dual foliations to show that the Sharafutdinov projection is smooth. For more applications of dual foliations, the reader is referred to [7,9–12,16].

E-mail address: shiyi@sxu.edu.cn.

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Let $(M, \mathcal{F})$ be a singular Riemannian foliation $\mathcal{F}$ on Riemannian manifold M. Applying Wilking's theorem, we can get the horizontal connectivity by assuming that M has positive curvature. However, in this paper we are interested in getting the horizontal connectivity by applying the methods of subriemannian geometry. For a more extensive exposition of subriemannian geometry, we refer the readers to [4,5,14].

Recall that a *subriemannian geometry* on a manifold M consists of a distribution, which is to say a vector subbundle $H \subset TM$ of the tangent bundle of M, together with a fiber inner-product on H. We call H the horizontal distribution. A curve on M is called horizontal if it is tangent to H.

Let H be the horizontal distribution on M, the Lie brackets of vector fields in H generate the flag

$$H = H^1 \subset H^2 \subset \cdots \subset H^r \subset \cdots \subset TM$$

with

$$H^{r+1} = H^r + [H, H^r] \;\text{ for }\; r \geq 1$$

where

$$[H, H^r] = span\{[X, Y] : X \in H, Y \in H^r\}$$

At a point $p \in M$, this flag gives a flag of subspaces of T_pM:

$$H_p = H_p^1 \subset H_p^2 \subset \cdots \subset H_p^r \subset \cdots \subset T_pM$$

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$$H_p = H_p^1 \subset H_p^2 \subset \cdots \subset H_p^r \subset \cdots \subset T_pM$$

We say that H is *bracket generating* at p if there is an $r \in \mathbb{Z}^+$ such that $H_p^r = T_pM$, and H is *bracket generating* if for all $x \in M$ there is an $r(x) \in \mathbb{Z}^+$ such that $H_x^{r(x)} = T_xM$. The smallest integer r such that $H_p^r = T_pM$ is called the *step* of the distribution at p.

Chow's Theorem *([6]). Let M be a connected manifold and $H \in TM$ be a bracket generating distribution, then the set of points that can be connected to $p \in M$ by a horizontal path coincides with M.*

When Chow's condition fails on some subset of M, sometimes the horizontal connectivity also fails ([5], p. 82).

Suppose a compact connected Lie group G acts isometrically on manifold M. Throughout the paper, every action will be assumed to be *effective*. It is well known that the action of G induces a singular Riemannian foliation $(M, \mathcal{F})$ on M. Let M' denote the union of all principal orbits in M. An basic fact is that the restricted foliation $(M', \mathcal{F}|_{M'})$ is a regular Riemannian foliation on M'. Denote by H' the collection of horizontal spaces H_x with $x \in M'$, then H' is the horizontal distribution of $\mathcal{F}|_{M'}$ on M'. Contrast to Chow's Theorem, to connect any two points on M by a horizontal curve, we only need to assume that H' is bracket-generating at one point:

Lemma 1.1. *Let G be a compact connected Lie group, acting isometrically on a complete Riemannian manifold M. If H' is bracket-generating at some point $p \in M'$, then the dual foliation has only one leaf. If M is compact, then there exists a constant $C = C(M, G)$ such that any two points of M can be connected by a horizontal curve of length $\leq C$.*