Differential Geometry and its Applications 51 (2017) 117–152

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Pure spinors, intrinsic torsion and curvature in odd dimensions

Arman Taghavi-Chabert

Masaryk University, Faculty of Science, Department of Mathematics and Statistics, Kotlářská 2, 611 37 Brno, Czech Republic

A R T I C L E I N F O

Article history:
Received 3 February 2014
Received in revised form 16 February 2017
Communicated by M.G. Eastwood

Keywords:
Complex Riemannian geometry
Pure spinors
Distributions
Intrinsic torsion
Curvature prescription
Spinorial equations

A B S T R A C T

We study the geometric properties of a $(2m + 1)$-dimensional complex manifold $\mathcal{M}$ admitting a holomorphic reduction of the frame bundle to the structure group $P \subset \mathrm{Spin}(2m+1, \mathbb{C})$, the stabiliser of the line spanned by a pure spinor at a point. Geometrically, $\mathcal{M}$ is endowed with a holomorphic metric g, a holomorphic volume form, a spin structure compatible with g, and a holomorphic pure spinor field ξ up to scale. The defining property of ξ is that it determines an almost null structure, i.e. an m-plane distribution $\mathcal{N}_\xi$ along which g is totally degenerate.

We develop a spinor calculus, by means of which we encode the geometric properties of $\mathcal{N}_\xi$ and of its rank-$(m + 1)$ orthogonal complement $\mathcal{N}_\xi^\perp$ corresponding to the algebraic properties of the intrinsic torsion of the P-structure. This is the failure of the Levi-Civita connection ∇ of g to be compatible with the P-structure. In a similar way, we examine the algebraic properties of the curvature of ∇.

Applications to spinorial differential equations are given. Notably, we relate the integrability properties of $\mathcal{N}_\xi$ and $\mathcal{N}_\xi^\perp$ to the existence of solutions of odd-dimensional versions of the zero-rest-mass field equation. We give necessary and sufficient conditions for the almost null structure associated to a pure conformal Killing spinor to be integrable. Finally, we conjecture a Goldberg–Sachs-type theorem on the existence of a certain class of almost null structures when $(\mathcal{M}, g)$ has prescribed curvature.

We discuss applications of this work to the study of real pseudo-Riemannian manifolds.

1. Introduction and motivation

The present article is the odd-dimensional counterpart of the author's work presented in [39]. Both articles work share the same motivations and goals, and the reader should refer to the latter work for further details.

Let $(\mathcal{M}, g)$ be an n-dimensional complex Riemannian manifold, where $n = 2m + 1$. We shall assume that $(\mathcal{M}, g)$ is also equipped with a global holomorphic volume form and a holomorphic spin structure so that the structure group of the holomorphic frame bundle is reduced to $G := \mathrm{Spin}(n, \mathbb{C})$. We work in the holomorphic category. We shall be considering a *projective pure spinor field* $[\xi]$, i.e. a spinor field up to

E-mail address: taghavia@math.muni.cz.

http://dx.doi.org/10.1016/j.difgeo.2017.02.008
0926-2245/

scale that annihilates a totally null m-plane, or γ-*plane*, distribution. This will also be referred to as its associated *almost null structure* $\mathcal{N}_\xi$. The structure group of the frame bundle of $(\mathcal{M}, g)$ is reduced to P, the stabiliser of $[\xi]$ at a point. Denote by $\mathfrak{g}$ and $\mathfrak{p}$ the respective Lie algebras of G and P, and by $\mathfrak{V}$ the standard representation of $\mathfrak{g}$. The main aim of the article is to examine the geometric properties of the P-structure on $(\mathcal{M}, g)$. More specifically, we will

- give a P-invariant decomposition of the space $\mathfrak{W} := \mathfrak{V} \otimes (\mathfrak{g}/\mathfrak{p})$ of intrinsic torsions;
- give P-invariant decompositions of the spaces of curvature tensors, in particular, tracefree Ricci tensors, Cotton–York tensors and Weyl tensors;
- apply these decompositions to the study of almost null structures and pure spinor fields on complex Riemannian manifolds.

The methodology will be a synthesis of representation theory and a spinor calculus adapted to the P-structure. Before we proceed, we first highlight the crucial differences between the odd- and even-dimensional cases:

- there is only one irreducible spinor representation of G as opposed to two chiral ones – paradoxically, this makes the spinor calculus more fiddly;
- the stabiliser $\mathfrak{p}$ of $[\xi]$ induces a $|2|$-grading on $\mathfrak{g}$, rather than a $|1|$-grading;
- the orthogonal complement $\mathcal{N}_\xi^\perp$ of $\mathcal{N}_\xi$ is $(m+1)$-dimensional and contains $\mathcal{N}_\xi$, rather than $\mathcal{N}_\xi^\perp = \mathcal{N}_\xi$.

Consequently, one has to encode the properties of *both* $\mathcal{N}_\xi$ and $\mathcal{N}_\xi^\perp$ in terms of differential conditions on $[\xi]$, although there is some degree of interdependency between $\mathcal{N}_\xi$ and $\mathcal{N}_\xi^\perp$. Making the move from even to odd dimensions is therefore not always straightforward. A case in point is when $\mathcal{N}_\xi$ is integrable. In even dimensions, $\mathcal{N}_\xi$ would be automatically totally geodetic, but in odd dimensions, this condition is stronger. In addition, one could have the extra requirement for $\mathcal{N}_\xi^\perp$ to be also integrable, and or even totally geodetic. This is particularly relevant to generalisations of the Robinson theorem, which can be strikingly different.

The present article can, if not should, be read in conjunction with [39] for comparison and ease of understanding of the notions introduced in the latter. Indeed, these two papers are broadly 'mirror images' of each other: the overall structure is the same in both papers as far as the numbering of the sections is concerned. For the sake of conciseness, we have not always deemed it necessary to re-establish notations and conventions.

Structure of the paper: Our spinor calculus will first be developed in section 2. New results include Propositions 2.6 and 2.9, and Corollary 2.10, which provide simpler alternatives to some of Cartan's formulae on pure spinors. Proposition 3.2 in section 3 is concerned with the decomposition of the space of intrinsic torsions of a P-structure. In the same vein, in section 4, Propositions 4.1, 4.2 and 4.4 give P-invariant decompositions of the spaces of tracefree Ricci tensors, Cotton–York tensors and Weyl tensors respectively.

Section 5 focuses on the geometric applications. Proposition 5.4 is the geometric articulation of Proposition 3.2. Proposition 5.7, Lemma 5.8 and Proposition 5.11 are concerned with geometric interpretations of $\mathcal{N}_\xi$ in terms of $\nabla[\xi]$. Three distinct generalisations of the Robinson theorems for three distinct types of zero-rest-mass fields are given in Theorems 5.19, 5.20 and 5.21. Applications to conformal Killing spinors are given in Propositions 5.24, 5.28 and 5.30. Conjecture 5.32 postulates a generalisation of the Goldberg–Sachs theorem given in [37]. Integrability conditions for solutions of the field equations involved are also given in Propositions 5.12, 5.13, 5.14, 5.17, 5.23 and 5.27 among others.

Appendix A contains useful formulae to characterise tracefree Ricci, Cotton–York and Weyl tensors in the light of the decompositions given in section 4. A brief discussion of spinor calculus in dimensions three