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Hamilton type gradient estimate for a nonlinear diffusion equation on smooth metric measure spaces

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ABSTRACT

In this note, we show an Hamilton type gradient estimate for the solution of time-dependent of nonlinear diffusion equation on smooth metric measure space $(M, g, e^{-f}dv)$. As the application, we show a dimension-free Harnack inequality and prove a Liouville property for f-harmonic functions.

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1. Introduction

Let $(M, g, e^{-f}dv)$ be an n-dimensional complete non-compact smooth metric measure space. On M, we consider the nonlinear diffusion equation

$$u_t = \Delta u - \langle \nabla f, \nabla u \rangle - au \ln u + hu, \tag{1.1}$$

where $a \in \mathcal{C}(M)$, $f \in \mathcal{C}^2(M)$, and $h = h(x,t)$ is a function defined on $M \times (0, \infty)$ which is $\mathcal{C}^1$ in the x-variable. When $f \equiv const$, the equation (1.1) is closely linked with the gradient Ricci solitons which are self-similar solutions to the Ricci flow introduced by R.S. Hamilton (see [6]).

Here we recall the definition of gradient Ricci solitons (see chapter 4 of [5]).

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Definition 1.1. A Riemannian manifold (M, g) is called a gradient Ricci soliton if there exists a smooth function $\varphi : M \to \mathbb{R}$, sometimes called potential function, such that for some constant $\lambda \in \mathbb{R}$, it satisfies

$$\mathrm{Ric}_g + \nabla^g \nabla^g \varphi = \lambda g \tag{1.2}$$

on M, where Ric_g is the Ricci curvature of manifold M and $\nabla^g \nabla^g \varphi$ is the Hessian of φ. A soliton is said to be shrinking, steady or expanding if the constant λ is respectively positive, zero or negative.

Assume that (M, g) is a gradient Ricci soliton as the above definition. Letting $u = e^{\varphi}$, under some curvature conditions, we can derive from (1.2) (see [11]) that

$$\Delta u + 2cu \ln u = (A_0 - n\lambda)u,$$

for some constant A_0. Hence, the study of the equation (1.1) is of interest (see [3,7] and references therein).

In the case of weighted Laplacian, namely $f \not\equiv const$, recently, X.D. Li (see [10]), Q.H. Ruan (see [12]), and other mathematicians, for example [8], studied the linear heat equation (1.1) on a noncompact Riemannian manifold with assuming $a \equiv 0$, $h \leq 0$ and the Bakry-Émery Ricci curvature. Li studied gradient estimates of Li–Yau type, Ruan investigated gradient estimates of Hamilton type.

In particular, Ruan proved the following theorem.

Theorem 1.2 *(Ruan). Let (M, g) be an n-dimensional complete noncompact manifold satisfying $Ric_{m,n} \geq -K, K \geq 0$ is constant. Suppose that $a = 0, h \leq 0$ and that u is any positive solution to the nonlinear diffusion equation (1.1) with $u \leq C$, for all $(x, t) \in M \times (0, \infty)$. Then*

$$\frac{|\nabla u|}{u} \leq \left(\frac{1}{t^{1/2}} + \sqrt{2K} + \left| \nabla \sqrt{-h} \right|^{1/2} \right) \left(1 + \ln \left(\frac{C}{u} \right) \right) \tag{1.3}$$

Recall that the m-dimensional Bakry-Émery curvature $\mathrm{Ric}_{m,n}$ is defined by

$$\mathrm{Ric}_{m,n} := \mathrm{Ric} + \nabla^2 f - \frac{\nabla f \otimes \nabla f}{m - n}.$$

When m tends to infinity, the m-dimensional Bakry-Émery becomes the Bakry-Émery curvature,

$$Ric_f := Ric + \nabla^2 f$$

Regarding to applications of equation (1.1) to study Ricci flow, we refer to [4,9] and references therein.

In this paper, we show a global Hamilton type gradient estimate for the positive solution of the nonlinear diffusion equation (1.1). Our main theorem can be stated as follows

Theorem 1.3 *(Main theorem). Let $(M, g, e^{-f} dv)$ be an n-dimensional complete noncompact smooth metric measure space satisfying $Ric_f \geq -K, K \geq 0$ is constant. Suppose that u is any positive solution to the nonlinear diffusion equation (1.1) with $u \leq C$ for all $(x, t) \in M \times (0, \infty)$.*

a) If $h - a \ln C - a \leq 0$ then

$$\frac{|\nabla u|}{u} \leq \frac{1}{\sqrt{2}} \left(\frac{1}{t^{1/2}} + \sqrt{(2 + C_2)K} + \sqrt{2} \left| \nabla \sqrt{-(h - a \ln C - a)} \right|^{1/2} \right) \left(1 + \ln \left(\frac{C}{u} \right) \right). \tag{1.4}$$

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