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A Szegö type theorem for truncated Toeplitz operators

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Abstract

Truncated Toeplitz operators are compressions of multiplication operators on L^2 to model spaces (that is, subspaces of H^2 which are invariant with respect to the backward shift). For this class of operators we prove certain Szegö type theorems concerning the asymptotics of their compressions to an increasing chain of finite dimensional model spaces.

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The Toeplitz operators are compressions of multiplication operators on the space $L^2(\mathbb{T})$ to the Hardy space H^2; the multiplier is called the symbol of the operator. With respect to the standard exponential basis, their matrices are constant along diagonals; if we truncate such a matrix considering only its upper left finite corner, we obtain classical Toeplitz matrices.

It does not come as a surprise that there are connections between the asymptotics of these Toeplitz matrices and the whole Toeplitz operator, or its symbol. A central result is Szegö's strong limit theorem and its variants (see, for instance, [2] and the references within), which deal

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with the asymptotics of the eigenvalues of the Toeplitz matrix. The version that is a starting point for our investigation states that, if T_n are truncated Toeplitz matrices corresponding to the symbol a, then

$$\frac{1}{n}\operatorname{Tr} T_n^p \to \int_{\mathbb{T}} a^p \, dm.$$

On the other hand, certain generalizations of Toeplitz matrices have attracted a great deal of attention in the last decade, namely compressions of multiplication operators to subspaces of the Hardy space which are invariant under the backward shift. These "model spaces" are of the form $H^2 \ominus uH^2$ with u an inner function, and the compressions are called truncated Toeplitz operators. They have been formally introduced in [10]; see [6] for a more recent survey. Although classical Toeplitz matrices have often been a starting point for investigating truncated Toeplitz operators, the latter may exhibit surprising properties.

It thus seems natural to see whether an analogue of Szegö's strong limit theorem can be obtained in this more general context. Viewed as truncated Toeplitz operators, the Toeplitz matrices act on model spaces corresponding to the inner functions $u(z) = z^n$, and Szegö's theorem is about the asymptotical situation when $n \to \infty$. The natural generalization is then to consider a sequence of zeros (λ_j) in $\mathbb{D}$, and to let the truncations act on the model space corresponding to the finite Blaschke product associated to λ_j, $1 \le j \le n$.

Such a result has been obtained in [1]; it deals with the asymptotics of the determinant of a truncated Toeplitz operator. Let us note that in the case of classical Toeplitz operators and matrices one has different variants of Szegö's Theorem, either in terms of the determinant of the truncation, or in terms of the trace, and one can pass from one to the other. However, this is no longer true in our generalized context, where the two different classes of results do not have a visible connection.

The purpose of this paper is to find an analogue of the trace type Szegö theorem. We manage to obtain a complete result in the case when (λ_j) is not a Blaschke sequence. The Blaschke case seems to be less prone to an elegant solution, and we have only partial conclusions.

The technique we use is inspired by one of the approaches to the Szegö Theorem that is based on approximation by circulants (see, for instance, [8]). In our case we use the analogue of circulants for truncated Toeplitz operators, namely elements in the so-called Sedlock algebras [11].

The plan of the paper is the following. After a rather extensive preliminary section that introduces the basic notions, we discuss the special case of finite dimensional model spaces. We introduce then the Sedlock algebras in Section 3, and prove a result important in its own right, Theorem 3.2, which gives an alternate identification of these algebras. Sedlock algebras on finite dimensional model spaces are briefly discussed in Section 4, after which Section 5 develops the approximation technique based on them. The main result, Theorem 6.1, is proved in Section 6. The last section discusses through some examples the problems that appear when considering Blaschke sequences.

1. Preliminaries

1.1. Model spaces

Let H^2 be the Hardy space of square integrable functions on the circle with negative Fourier coefficients equal to 0. We recall that a *model* space is a subspace of H^2 which is invariant for the backwards shift, and that every such space is of the form $K_B = H^2 \ominus BH^2$ where B is an inner