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Orthogonal polynomial projection error measured in Sobolev norms in the unit ball

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Abstract

We study approximation properties of weighted L^2-orthogonal projectors onto spaces of polynomials of bounded degree in the Euclidean unit ball, where the weight is of the generalized Gegenbauer form $x \mapsto (1 - \|x\|^2)^\alpha$, $\alpha > -1$. Said properties are measured in Sobolev-type norms in which the same weighted L^2 norm is used to control all the involved weak derivatives. The method of proof does not rely on any particular basis of orthogonal polynomials, which allows for a short, streamlined and dimension-independent exposition.
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1. Introduction

It has been known since the early eighties [5] that the orthogonal projector S_N^0 mapping $L^2(-1, 1)$ onto the space of univariate polynomials of degree less than or equal to N (equivalently, S_N^0 is the operation consisting in truncating the Fourier–Legendre series of its

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argument at degree N) satisfies the bound

$$(\forall u \in \mathrm{H}^l(-1,1)) \quad \|u - S_N^0(u)\|_{\mathrm{H}^1(-1,1)} \leq C N^{3/2-l} \|u\|_{\mathrm{H}^l(-1,1)}, \tag{1}$$

where $C > 0$ depends only on l and $\mathrm{H}^1(-1,1)$ and $\mathrm{H}^l(-1,1)$ denote standard Sobolev spaces (see [4, Ch. 5] for a detailed proof of (1) and its Chebyshev weight and periodic unweighted analogues and [11] for its general Gegenbauer weight analogue). Recently [10] this result was extended to the unit disk for Gegenbauer-type weights.

The purpose of this work is proving a weighted analogue of (1) in the case of the unit ball of any dimension; in order to state it, we introduce now the minimal necessary notation. Let B^d be the unit ball of $\mathbb{R}^d$, $\alpha > -1$ and let the weight function $W_\alpha : B^d \to \mathbb{R}$ be defined by $W_\alpha(x) = (1 - \|x\|^2)^\alpha$, with $\|\cdot\|$ being the Euclidean norm. We denote by L^2_α the weighted Lebesgue space $\mathrm{L}^2(B^d, W_\alpha) := \{W_\alpha^{-1/2} f \mid f \in \mathrm{L}^2(B^d)\}$, whose natural squared norm is $\|u\|_\alpha^2 := \int_{B^d} |u|^2 W_\alpha$; per the polarization identity its inner product is $\langle u, v\rangle_\alpha = \int_{B^d} u\,\overline{v}\, W_\alpha$. Given an integer $l \geq 0$, we denote by H^l_α the weighted Sobolev space whose squared norm is $\|u\|^2_{\mathrm{H}^l_\alpha} := \sum_{k=0}^{l} \|\nabla_k u\|_\alpha^2$. Let S_N^α be the orthogonal projector mapping L^2_α onto Π_N^d, where Π_N^d is the space of d-variate polynomials of degree less than or equal to N. Our main result is

Theorem 1.1. *For all integers $1 \leq r \leq l$ there exists $C = C(d, \alpha, l, r) > 0$ such that*

$$(\forall u \in \mathrm{H}^l_\alpha) \quad \|u - S_N^\alpha(u)\|_{\mathrm{H}^r_\alpha} \leq C N^{-1/2+2r-l} \|u\|_{\mathrm{H}^l_\alpha}. \tag{2}$$

There are two application domains of our main result that we are aware of. One lies in the analysis of polynomial interpolation operators (cf. [5] and [4, Ch. 5]), themselves important in the analysis of spectral methods. The other, which is the one that led us into this pursuit in the first place, lies in the characterization of approximability spaces relevant to the analysis of nonlinear iterative methods for the numerical solution of high-dimensional PDE; we remit the interested reader to [9, Ch. 4] where the one-dimensional case of Theorem 1.1 and the fact that the S_N^α projectors tensorize in a very straightforward way are exploited for such task.

The behavior of S_N^α—the weighted L^2 projector onto algebraic polynomials over the unit ball—expressed in Theorem 1.1 compares unfavorably with that of the unweighted L^2 projector onto spherical harmonics over the unit sphere of $\mathbb{R}^d$, $d \geq 2$ [15, Th. 2.8] (there the involved Sobolev spaces over the unit sphere are also allowed to be based upon general L^p spaces, $1 < p < \infty$), where, using our notation, the power on the degree N is simply $r - l$. In the particular case of the unit circle, the $p = 2$ case of [15, Th. 2.8] can be recast as a statement on the behavior of the unweighted L^2 projector onto trigonometric polynomials over $(0, 2\pi)$ involving unweighted periodic Sobolev spaces [4, Section 5.1]. The origin of this difference of behavior is that in the spherical harmonics and trigonometric cases the relevant differentiation and projection operators commute, something which is impossible in our setting involving both algebraic polynomials and standard differentiation operators [4, Section 2.3.2].

We emphasize that the case $r = 0$ is explicitly excluded from consideration in Theorem 1.1, for in such a case the provably optimal power on N is $-l$ (cf. Lemma 2.3), outside the pattern set in (2). We also note that if $2r \geq l + 1/2$ in (2), $S_N^\alpha(u)$ need not converge to u in H^r_α as N tends to infinity. We further remark that Theorem 1.1 is not a best or quasi-best approximation result (for those see [4, Ch. 5], [11], [15, Section 4] and [6, Section 5]), because in general the orthogonal projection of H^r_α onto Π_N^d need not coincide with the restriction of S_N^α to H^r_α.

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