Available online at www.sciencedirect.com
ScienceDirect
Journal of Approximation Theory 223 (2017) 1–8
Journal of Approximation Theory
www.elsevier.com/locate/jat

Full Length Article

Interpolatory estimates in monotone piecewise polynomial approximation

D. Leviatan[a,*], I.L. Petrova[b]

[a] Raymond and Beverly Sackler School of Mathematical Sciences, Tel Aviv University, Tel Aviv 69978, Israel
[b] Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, 01601 Kyiv, Ukraine

Received 29 August 2016; received in revised form 25 May 2017; accepted 7 July 2017
Available online 20 July 2017

Communicated by: Paul Nevai

Abstract

Given a monotone function $f \in C^r[-1,1]$, $r \ge 1$, we obtain pointwise estimates for its monotone approximation by piecewise polynomials involving the second order modulus of smoothness of $f^{(r)}$. These estimates are interpolatory estimates, namely, the piecewise polynomials interpolate the function at the endpoints of the interval. However, they are valid only for $n \ge N(f,r)$. We also show that such estimates are in general invalid with N independent of f.

MSC: 41A10; 41A25; 41A29; 41A30

Keywords: Monotone approximation by piecewise polynomials; Degree of pointwise approximation; Jackson-type interpolatory estimates

1. Introduction and the main result

For $r \in \mathbb{N}$, let $C^r[a,b]$, $-1 \le a < b \le 1$, denote the space of r times continuously differentiable functions on $[a,b]$, and let $C^0[a,b] = C[a,b]$ denote the space of continuous functions on $[a,b]$, equipped with the uniform norm $\|\cdot\|_{[a,b]}$. When dealing with $[-1,1]$, we omit the reference to the interval, that is, we denote $\|\cdot\| := \|\cdot\|_{[-1,1]}$. Let $\mathbb{P}_n$ be the space of algebraic polynomials of degree $\le n$.

* Corresponding author.
E-mail addresses: leviatan@post.tau.ac.il (D. Leviatan), irynapetrova1411@gmail.com (I.L. Petrova).

http://dx.doi.org/10.1016/j.jat.2017.07.006
0021-9045/

For $f \in C[a,b]$ and any $k \in \mathbb{N}$, set

$$\Delta_u^k(f,x;[a,b]) := \begin{cases} \sum_{i=0}^{k}(-1)^i \binom{k}{i} f(x+(k/2-i)u), & x \pm (k/2)u \in [a,b] \\ 0, & \text{otherwise,} \end{cases}$$

and denote by

$$\omega_k(f,t;[a,b]) := \sup_{0<u\le t} \|\Delta_u^k(f,\cdot;[a,b])\|_{[a,b]},$$

its kth modulus of smoothness. For $[a,b]=[-1,1]$, write $\omega_k(f,t) := \omega_k(f,t;[-1,1])$.

Let $X_n := \{x_{j,n}\}_{j=0}^n$, $x_{j,n} = -\cos j\pi/n$, $0 \le j \le n$, be the Chebyshev partition of $[-1,1]$ (see, *e.g.*, [6]), and set $x_{n+1,n} := 1$, $x_{-1,n} := -1$.

Finally, let

$$\varphi(x) = \sqrt{1-x^2} \quad \text{and} \quad \rho_n(x) := \frac{\varphi(x)}{n} + \frac{1}{n^2}. \tag{1.1}$$

Pointwise estimates have mostly been investigated for polynomial approximation of continuous functions in $[-1,1]$ and involved usually the quantity $\rho_n(x)$. The first to deal with such estimates was Nikolskii, and he was followed by Timan, Dzjadyk, Freud and Brudnyi. Detailed discussion may be found in the survey paper [5], where an extensive list of references is given. Discussion and references to estimates on pointwise monotone polynomial approximation involving $\rho_n(x)$ also may be found there. Pointwise estimates of polynomial approximation involving $\varphi(x)$ are due originally to Teljakovskiĭ and Gopengauz, see [4] for extensions and many references. Finally, for some results on pointwise rational approximation, see [1].

The main result of this paper is the following.

Theorem 1.1. *Given $r \in \mathbb{N}$, there is a constant $c = c(r)$ with the property that if a function $f \in C^r[-1,1]$, is monotone, then there is a number $N = N(f,r)$, depending on f and r, such that for $n \ge N$, there are monotone continuous piecewise polynomials s of degree $r+1$ with knots at the Chebyshev partition, satisfying*

$$|f(x)-s(x)| \le c(r)\left(\frac{\varphi(x)}{n}\right)^r \omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1,1], \tag{1.2}$$

and

$$|f(x)-s(x)| \le c(r)\varphi^{2r}(x)\omega_2\left(f^{(r)}, \frac{\varphi(x)}{n}\right), \quad x \in [-1,x_{1,n}] \cup [x_{n-1,n},1]. \tag{1.3}$$

Remark 1.2. Theorem 1.1 is well known for $r=0$, in fact, with $N=1$. Indeed, the polygonal line, that is, the continuous piecewise linear s, interpolating f at the Chebyshev nodes, is nondecreasing and yields (1.2) with $r=0$ (see, *e.g.*, a similar construction in [2]). It may be worth mentioning that if f is convex, then the same polygonal line is convex, thus we have the estimate (1.2) with $r=0$ also for convex approximation. This was shown in [7].

In the sequel all constants c will depend on r, but may otherwise be different in each occurrence.

2. Monotone approximation by piecewise polynomials on a general partition

Given $[a,b]$, let $X = \{x_j\}_{j=0}^n$ be a partition of the interval, such that

$$a =: x_0 < x_1 < \cdots < x_n := b.$$