The growth of polynomials outside of a compact set –the Bernstein-Walsh inequality revisited

Klaus Schiefermayr

PII: S0021-9045(17)30089-8
DOI: http://dx.doi.org/10.1016/j.jat.2017.07.007
Reference: YJATH 5163

To appear in: *Journal of Approximation Theory*

Received date : 31 March 2017
Revised date : 27 June 2017
Accepted date : 24 July 2017

The Growth of Polynomials Outside of a Compact Set – the Bernstein-Walsh Inequality Revisited

Klaus Schiefermayr*

Abstract

In this paper, we present a new and simple proof of the classical Bernstein-Walsh inequality. Based on this proof, we give some improvements for this inequality in the case that the corresponding compact set is real.

Mathematics Subject Classification: 30C10, 30E10, 41A17, 41A50, 30C85
Keywords: Bernstein-Walsh inequality, Green function, Logarithmic capacity, Inverse polynomial image

1 The Bernstein-Walsh Inequality

Let K be a compact set in the complex plane $\mathbb{C}$ with logarithmic capacity $\operatorname{cap} K > 0$. Without loss of generality, we assume that K is such that $\overline{\mathbb{C}} \setminus K$ is connected, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the extended complex plane. Let $g_K(z)$ denote the Green function (with pole at ∞) for $\overline{\mathbb{C}} \setminus K$ and define $g_K(z) := 0$ for $z \in K$. Furthermore, let $\mathbb{P}_n$ denote the set of all polynomials of degree n with complex coefficients and let $\|\cdot\|_K$ denote the supremum norm on K. Then the *Bernstein-Walsh inequality* (sometimes called *Bernstein-Walsh lemma* or *Bernstein lemma*), see [8, Theorem 5.5.7] or [11, Lemma 3.7] or [7, Section 12.1], reads as follows.

Theorem 1 (Bernstein-Walsh inequality)**.** *For any polynomial $Q_n \in \mathbb{P}_n$,*

$$\frac{|Q_n(z)|}{\|Q_n\|_K} \leq \mathrm{e}^{n \cdot g_K(z)} \qquad (z \in \mathbb{C} \setminus K). \tag{1}$$

Note that inequality (1) is obviously also true for $z \in K$. Inequality (1) gives a very general upper bound for the modulus of a polynomial outside a compact set K with respect to its maximum value on K in terms of the corresponding Green function (which only depends on K). It is also common to consider inequality (1) with the notion of the level sets of the Green function $g_K(z)$. For each $R \geq 1$, let

$$\Gamma_R(K) := \bigl\{z \in \mathbb{C} : g_K(z) = \log R\bigr\} \tag{2}$$

define the *level set with index* R for the Green function $g_K(z)$ (note that $\Gamma_R(K) = K$ for $R = 1$). Then inequality (1) reads as

$$\frac{|Q_n(z)|}{\|Q_n\|_K} \leq R^n \qquad (z \in \Gamma_R(K)). \tag{3}$$

*University of Applied Sciences Upper Austria, Campus Wels, School of Engineering, Stelzhamerstrasse 23, 4600 Wels, Austria, KLAUS.SCHIEFERMAYR@FH-WELS.AT