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Approximation properties of combination of multivariate averages on Hardy spaces

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Abstract

In this paper, we study the rate of approximation of the combination of some generalized multivariate average on Hardy spaces and obtain its equivalent relation to the K-functionals. The result is an extension of a result in Dai and Ditzian (2004). We also extend and improve Theorem 6.2 in Belinsky et al. (2003).

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1. Introduction

Let $\gamma \in \mathbb{R}$ and let I_γ be the Riesz potential of order γ defined on functions or distributions g via the Fourier transform

$$\widehat{I_\gamma(g)}(\xi) = |\xi|^{-\gamma}\widehat{g}(\xi).$$

The Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ on the n-dimensional Euclidean space $\mathbb{R}^n$ satisfies $\Delta = -I_{-2}$.

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Fix a Schwartz function Φ satisfying

$$\int_{\mathbb{R}^n} \Phi(x)dx \neq 0.$$

The Hardy space $H^p(\mathbb{R}^n)$, $0 < p < \infty$, is the space of all distributions f satisfying

$$\|f\|_{H^p(\mathbb{R}^n)} = \left\| \sup_{t>0} |\Phi_t * f| \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where $\Phi_t(y) = t^{-n}\Phi(y/t)$ for $t > 0$. The space $H^p(\mathbb{R}^n)$ is a quasi-Banach space for any $0 < p < 1$ and is a Banach space if $p \geq 1$. Particularly, we know that $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ if $1 < p < \infty$. An important characterization of $H^p(\mathbb{R}^n)$, when $0 < p \leq 1$, is that it can be defined by using the Riesz transforms. For an integer $L > 0$, and a multi-index $J = \{j_1, \ldots, j_L) \in \{0, 1, 2, \ldots, n\}^L$, let $R_J(f)$ denote the generalized Riesz transform $R_J(f) = R_{j_1} \ldots R_{j_L}(f)$, where $R_j(f)$ is the jth Riesz transform of f if $j \neq 0$ and $R_0(f) = f$. It is known in [9, pp. 167–168] that for $p > \frac{n-1}{n-1+L}$ and all $f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$

$$\|f\|_{H^p(\mathbb{R}^n)} \approx \sum_{J\in\{0,1,2,\ldots,n\}^L} \|R_J(f)\|_{L^p(\mathbb{R}^n)}.$$

Hereafter, the notation $A \preceq B$ means that there is a positive constant C independent of all essential variables such that $A \leq CB$. The notation $A \approx B$ means that there are two positive constants C_1 and C_2 independent of all essential variables such that $C_1 A \leq B \leq C_2 A$.

Suppose that $t > 0$, $\gamma > 0$, $0 < p < \infty$ and $f \in H^p(\mathbb{R}^n)$. Let us denote by $K_\gamma(f, t)_{H^p(\mathbb{R}^n)}$ the γth order K-functional of f, that is,

$$K_\gamma(f, t)_{H^p(\mathbb{R}^n)} = \inf_{g\in H^{p,\gamma}(\mathbb{R}^n)} \left\{ \|f - g\|_{H^p(\mathbb{R}^n)} + t^\gamma \left\|I_{-\gamma}(g)\right\|_{H^p(\mathbb{R}^n)} \right\},$$

where

$$H^{p,\gamma}(\mathbb{R}^n) = \left\{ g \in H^p(\mathbb{R}^n) : \ I_{-\gamma}(g) \in H^p(\mathbb{R}^n) \right\},$$

and we recall that $I_{-\gamma}(g)$ is defined through its Fourier transform

$$\widehat{I_{-\gamma}(g)}(\xi) = |\xi|^\gamma \widehat{g}(\xi).$$

Similarly, for $t > 0$, $\gamma > 0$ and $1 \leq p \leq \infty$, we use the symbol $K_\gamma(f, t)_{L^p(\mathbb{R}^n)}$ to denote the γth order K-functional of f

$$K_\gamma(f, t)_{L^p(\mathbb{R}^n)} = \inf_{g\in L^{p,\gamma}(\mathbb{R}^n)} \left\{ \|f - g\|_{L^p(\mathbb{R}^n)} + t^\gamma \left\|I_{-\gamma}(g)\right\|_{L^p(\mathbb{R}^n)} \right\},$$

where

$$L^{p,\gamma}(\mathbb{R}^n) = \left\{ g \in L^p(\mathbb{R}^n) : \ I_{-\gamma}(g) \in L^p(\mathbb{R}^n) \right\}.$$

The K-functional of f, $K_\gamma(f, t)_{H^p}$ (resp. $K_\gamma(f, t)_{L^p}$), is used to measure the smoothness of f in H^p (resp. L^p) for different γ. Another measure, the modulus of smoothness of the mth order $\omega_m(f, t)_{H^p(\mathbb{R}^n)}$ is given by

$$\omega_m(f, t)_{H^p(\mathbb{R}^n)} = \sup_{|h|\leq t} \left\| \sum_{j=0}^{m} (-1)^j \binom{m}{j} f(\cdot + jh) \right\|_{H^p(\mathbb{R}^n)}$$

where m is a nonnegative integer. If we replace the Riesz potential $I_{-\gamma}$ with the Bessel potential $\mathcal{J}_{-\gamma}$ which is defined by Fourier transform $\widehat{\mathcal{J}_{-\gamma}(f)}(\xi) = (1 + |\xi|^2)^{\gamma/2}\hat{f}(\xi)$, then Colzani