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Abstract

Minkowski's question mark function is the distribution function of a singular continuous measure: we study this measure from the point of view of logarithmic potential theory and orthogonal polynomials. We conjecture that it is regular, in the sense of Ullman–Saff–Stahl–Totik and moreover that it belongs to a Nevai class; we provide numerical evidence of the validity of these conjectures. In addition, we study the zeros of its orthogonal polynomials and the associated Christoffel functions, for which asymptotic formulae are derived. As a by–product, we compute upper and lower bounds to the Hausdorff dimension of Minkowski's measure. Rigorous results and numerical techniques are based upon Iterated Function Systems composed of Möbius maps.

Keywords: Minkowski's question mark function; Orthogonal polynomials, Jacobi matrices; Regular Measures; Möbius Iterated Function Systems; Nevai class; Gaussian integration; Christoffel functions.
MATH Subj. Class. 42C05; 47B36; 11A55; 11B57; 37D40

1 Introduction: Minkowski's Q function and its singular measure

1.1 Theoretical setting and goals of the paper

Minkowski's question–mark function $Q(x)$ can be concisely defined—though not in the the most transparent way—by writing the point $x \in [0,1]$ in its continued fraction representation, $x = [n_1, n_2, \ldots,]$, by setting $N_j(x) = \sum_{l=1}^{j} n_l$, and by defining $Q(x)$ as the sum of the series [18, 52]

$$Q(x) = \sum_{j=1}^{\infty} (-1)^{j+1} 2^{-N_j(x)+1}. \tag{1}$$

This function was originally constructed to map the rationals to the solutions of quadratic equations with rational coefficients in a continuous, order preserving way [45], but it successively appeared that it has much wider implications in many fields of mathematics. A graph of $Q(x)$ is part of Figures 10 and 12 below. It is remarkable that this graph can be seen as the attractor of an *Iterated Function System* (IFS) composed of Möbius maps [14], so that $Q(x)$ also belongs to the family of fractal interpolation functions [12, 13].

In this paper we are interested in the singular-continuous measure μ of which $Q(x)$ is the distribution function:

$$Q(x) = \int_0^x d\mu. \tag{2}$$