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On approximation of ultraspherical polynomials in the oscillatory region

Ilia Krasikov

Department of Mathematics, Brunel University London, Uxbridge UB8 3PH, United Kingdom

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Abstract

For $k \geq 2$ even, and $\alpha \geq -(2k+1)/4$, we provide a uniform approximation of the ultraspherical polynomials $P_k^{(\alpha,\alpha)}(x)$ in the oscillatory region with a very explicit error term. In fact, our result covers all α for which the expression "oscillatory region" makes sense. To that end, we construct the almost equioscillating function $g(x) = c\sqrt{b(x)}\,(1-x^2)^{(\alpha+1)/2} P_k^{(\alpha,\alpha)}(x) = \cos \mathcal{B}(x) + r(x)$. Here the constant $c = c(k,\alpha)$ is defined by the normalization of $P_k^{(\alpha,\alpha)}(x)$, $\mathcal{B}(x) = \int_0^x b(x)dx$, and the functions $b(x)$ and $\mathcal{B}(x)$, as well as bounds on the error term $r(x)$, are given by some rather simple elementary functions.

MSC: 33C45

Keywords: Orthogonal polynomials; Ultraspherical polynomials; Gegenbauer polynomials; Uniform approximation

1. Introduction

The ultraspherical polynomials we deal with in this paper will be convenient to define in terms of Jacobi polynomials as $P_k^{(\alpha,\alpha)}(x)$, where we choose the standard normalization for the last function. We will use the bold character $\mathbf{P}_k^{(\alpha,\alpha)}(x)$ to denote the orthonormal Jacobi polynomials. Since we are going to consider the case $\alpha \leq -1$ as well, let us notice that the right hand side of

E-mail address: mastiik@brunel.ac.uk.

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the formula

$$\| P_k^{(\alpha,\alpha)} \|_{L_2}^2 = \int_{-1}^{1} (1-x^2)^{\alpha} \left(P_k^{(\alpha,\alpha)}(x) \right)^2 dx = \frac{2^{2\alpha+1}\,\Gamma^2(k+\alpha+1)}{(2k+2\alpha+1)\,\Gamma(k+2\alpha+1)k!},$$

and therefore the orthonormal normalization, still make sense as far as $\alpha > -\frac{k+1}{2}$.

Here we will establish a uniform approximation of the ultraspherical polynomials in the oscillatory region with an explicit error term for a vast range of the parameter α, in fact, for all α for which the expression "oscillatory region" makes sense. A few standard formulas we are using in the sequel may be found e.g. in [12, sec. 4].

Generally, the oscillatory region is an interval containing all but maybe a few of the extreme zeros of a polynomial. In fact, besides the simplest case of the Chebyshev polynomials, there are no global asymptotic approximations of the classical orthogonal polynomials on the whole real axis. One has to split it into the oscillatory, transition and monotonicity regions, working with each of them separately. Theorem 8.22.9 in [12] related to the Hermite case may serve as a good example of this subdivision.

There are a number of known asymptotic approximations for the Jacobi polynomials under these or those restrictions on the parameters α and β, starting from the classical case $|\alpha|, |\beta| \le 1/2$ considered in Szegö's book [12], or, for example, more recent results with asymptotically constant ratios of α/k and β/k (see e.g. [11,13] and references therein). However, if one is interested in uniform bounds, the situation becomes less studied, and we refer to the recent preprint [6] and the references therein for a review of known results.

To simplify otherwise complicated expressions in the sequel we introduce the following new parameters:

$$u = (k+\alpha)(k+\alpha+1), \quad q = (\alpha^2-1)/u, \tag{1}$$

which turn out to be quite natural in this context.

We start with the normal form of the differential equation for the ultraspherical polynomials

$$y'' + b^2 y = 0, \quad y = (1-x^2)^{(\alpha+1)/2} P_k^{(\alpha,\alpha)}(x), \tag{2}$$

where

$$b = b(x) = \frac{\sqrt{(1-q-x^2)u}}{1-x^2}, \tag{3}$$

and define the function

$$g(x) = \sqrt{b(x)}\, y(x), \tag{4}$$

that, as we will show, almost equioscillates in the interval $|x| < \sqrt{1-q}$.

Our main result is the following Theorem 1 that provides a uniform approximation of $g(x)$ for k even in the oscillatory region with a very explicit error term. The corresponding result for k odd may be readily obtained from e.g. the three term recurrence. To simplify the statement of the theorem involving multivalued functions, without loss of generality we will restrict ourselves to the case $x \ge 0$.

Theorem 1. *Let $k \ge 2$ be even and let x belong to one of the following intervals depending on the value of α:*

(i) $0 \le x \le \sqrt{1-\frac{1}{u}}, \quad |\alpha| < \sqrt{\frac{7}{6}}$;

(ii) $0 \le x \le \sqrt{1-q}, \quad \alpha \in \left[-\frac{2k+1}{4}, -\sqrt{\frac{7}{6}}\,\right] \bigcup \left[\sqrt{\frac{7}{6}}, \infty\right)$.