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A NOTE ON WALL'S MODIFICATION OF THE SCHUR ALGORITHM AND LINEAR PENCILS OF JACOBI MATRICES

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ABSTRACT. In this note we revive a transformation that was introduced by H. S. Wall and that establishes a one-to-one correspondence between continued fraction representations of Schur, Carathéodory, and Nevanlinna functions. This transformation can be considered as an analog of the Szegő mapping but it is based on the Cayley transform, which relates the upper half-plane to the unit disc. For example, it will be shown that, when applying the Wall transformation, instead of OPRL, we get a sequence of orthogonal rational functions that satisfy three-term recurrence relation of the form $(H-\lambda J)u = 0$, where u is a semi-infinite vector, whose entries are the rational functions. Besides, J and H are Hermitian Jacobi matrices for which a version of the Denisov-Rakhmanov theorem holds true. Finally we will demonstrate how pseudo-Jacobi polynomials (aka Routh-Romanovski polynomials) fit into the picture.

1. Introduction

In September of the year 1916, I. Schur submitted the first paper of the series of two [24], [25] that presented a new parametrization of functions that are analytic and bounded by 1 in the open unit disc $\mathbb{D}$ and an algorithm for computing the corresponding parameters. The algorithm is now known as the Schur algorithm. In fact, it's been literally a hundred years, and yet there is still a continuing interest in further developing the findings of I. Schur. One of the reasons for that is because the Schur algorithm is a successor of the Euclidean algorithm, which has many theoretical and practical applications. Another one is that the Schur algorithm is intimately related to orthogonal polynomials on the unit circle (hereafter abbreviated by OPUC) and the latter has seen an enormous progress since the beginning of the 21st century. However, the ideology of this note is based on an old result that appeared in 1944 in a paper by H. S. Wall [30]. Nevertheless, a proper recasting of the result gives new insights and perspectives to the theory of orthogonal polynomials. We will see it later but now let's briefly recall basics of the Schur algorithm. First of all, we need to consider a Schur function f, which is an analytic function mapping $\mathbb{D}$ to its closure $\overline{\mathbb{D}}$, that is,

$$\sup_{z\in\mathbb{D}} |f(z)| \leq 1.$$

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