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Dimension hopping and families of strictly positive definite zonal basis functions on spheres

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Abstract

Positive definite functions of compact support are widely used for radial basis function approximation as well as for estimation of spatial processes in geostatistics. Several constructions of such functions for $\mathbb{R}^d$ are based upon recurrence operators. These map functions of such type in a given space dimension onto similar ones in a space of lower or higher dimension. We provide analogs of these dimension hopping operators for positive definite, and strictly positive definite, zonal functions on the sphere. These operators are then used to provide new families of strictly positive definite functions with local support on the sphere.

Keywords: Zonal functions; Positive definite; Dimension hopping operators

1. Introduction

This paper investigates certain dimension hopping operators on spheres that preserve strict and non-strict positive definiteness of zonal functions. The operators are the analogs for the sphere

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of the dimension hopping *montée* and *descente* operators of Matheron [13] for radial functions $f : \mathbb{R}^d \to \mathbb{R}$. These latter operators were later rediscovered by Schaback and Wu [15]. Using the *montée* operator for the sphere, and some known strictly positive definite, zonal functions, we construct further families of locally supported, strictly positive definite zonal functions. For the purposes of computation it is useful that these new functions can be evaluated at a relatively low computational cost rather than being given by infinite series. The first construction is an analog for the sphere of the Wendland family [19] of radial basis functions for $\mathbb{R}^d$ starting from the function $A(x) = (1 - \|x\|_2)_+^{\lceil \frac{d+1}{2} \rceil}$ of Askey [2], which is strictly positive definite on $\mathbb{R}^d$. The unit sphere in $\mathbb{R}^{d+1}$ will be denoted by $\mathbb{S}^d$. Later in the paper a relationship is established between convolutions of zonal functions on $\mathbb{S}^{d+2}$ and those for related zonal functions on $\mathbb{S}^d$. This enables the construction of families of locally supported strictly positive definite zonal functions, essentially by the self-convolution of the characteristic functions of spherical caps. This is the analog for the sphere of the construction of the circular and spherical covariances for $\mathbb{R}^2$ and $\mathbb{R}^3$, and more generally of the construction of Euclid's hat functions (see Wu [20] and Gneiting [9]). See Ziegel [22] and the references there for related work in the statistics community on generating positive definite functions on the sphere via self-convolution.

In what follows let $\theta(x, y) = \arccos(x^T y)$ denote the geodesic distance on $\mathbb{S}^d$.

Definition 1.1. A continuous function $g : [0, \pi] \to \mathbb{R}$ is (zonal) positive definite on the sphere $\mathbb{S}^d$ if for all distinct point sets $X = \{x_1, \ldots, x_n\}$ on the sphere and all $n \in \mathbb{N}$, the matrices $M_X := \left[g(\theta(x_i, x_j))\right]_{i,j=1}^n$ are positive semi-definite, that is, $\boldsymbol{c}^T M_X \boldsymbol{c} \geq 0$ for all $\boldsymbol{c} \in \mathbb{R}^n$. The function g is *(zonal) strictly positive definite* on $\mathbb{S}^d$ if the matrices are all positive definite, that is, $\boldsymbol{c}^T M_X \boldsymbol{c} > 0$, for all nonzero $\boldsymbol{c} \in \mathbb{R}^n$. The notation Ψ_d will denote the cone of all positive definite functions on $\mathbb{S}^d$ and Ψ_d^+ the subcone of all strictly positive definite functions on $\mathbb{S}^d$. Λ_d will denote the cone of all functions $f \in C[-1, 1]$ such that $f \circ \cos \in \Psi_d$. Λ_d^+ will denote the cone of all functions $f \in C[-1, 1]$ such that $f \circ \cos \in \Psi_d^+$.

In what follows we abuse notation somewhat by referring to $f : [-1, 1] \to \mathbb{R}$ as a zonal function when it is $f \circ \cos : [0, \pi] \to \mathbb{R}$ which is the zonal function. In the same spirit we will refer to Λ_d and Λ_d^+ as cones of positive definite and strictly positive definite functions, even though strictly speaking the relevant cones consist of the zonal functions $\Lambda_d \circ \cos$ and $\Lambda_d^+ \circ \cos$.

Zonal positive definite functions (radial basis functions on the sphere) have been used for interpolation or approximation of scattered data on the sphere (see [7,8] and the references therein). The standard model in this setting is a linear combination of translates (rotations) of the *zonal basis function*. Thus, the interpolation problem is

Problem 1.2. Given a zonal function g, n distinct points $x_i \in \mathbb{S}^d$ and n corresponding values $f_i \in \mathbb{R}$, find coefficients $c_j \in \mathbb{R}$ such that

$$s(x) = \sum_{j=1}^n c_j g(\theta(x, x_j)), \quad x \in \mathbb{S}^d,$$

satisfies

$$s(x_i) = f_i, \quad 1 \leq i \leq n.$$