ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Representations of hypergeometric functions for arbitrary parameter values and their use

D.B. Karp[a,b,∗], J.L. López[c]

[a] Far Eastern Federal University, Vladivostok, Russia
[b] Institute of Applied Mathematics FEBRAS, Vladivostok, Russia
[c] Dpto. de Ingeniería Matemática e Informática, Universidad Pública de Navarra and INAMAT, Navarra, Spain

Received 14 October 2016; received in revised form 1 March 2017; accepted 27 March 2017

Available online xxxx

Communicated by Leonid Golinskii

Abstract

Integral representations of hypergeometric functions proved to be a very useful tool for studying their properties. The purpose of this paper is twofold. First, we extend the known representations to arbitrary values of the parameters and show that the extended representations can be interpreted as examples of regularizations of integrals containing Meijer's G function. Second, we give new applications of both, known and extended representations. These include: inverse factorial series expansion for the Gauss type function, new information about zeros of the Bessel and Kummer type functions, connection with radial positive definite functions and generalizations of Luke's inequalities for the Kummer and Gauss type functions.

MSC: 33C20; 33C60; 33F05; 42A82; 65D20

Keywords: Generalized hypergeometric function; Meijer's G function; Integral representation; Radial positive definite function; Inverse factorial series; Hadamard finite part

∗ Corresponding author at: Far Eastern Federal University, Vladivostok, Russia.
E-mail addresses: dimkrp@gmail.com (D.B. Karp), jl.lopez@unavarra.es (J.L. López).

http://dx.doi.org/10.1016/j.jat.2017.03.004
0021-9045/

Please cite this article in press as: D.B. Karp, J.L. López, Representations of hypergeometric functions for arbitrary parameter values and their use, Journal of Approximation Theory (2017), http://dx.doi.org/10.1016/j.jat.2017.03.004

1. Introduction

Throughout the paper we will use the standard definition of the generalized hypergeometric function ${}_pF_q$ as the sum of the series

$$
{}_pF_q\left(\begin{matrix}\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| z\right) = {}_pF_q\,(\mathbf{a};\mathbf{b};z) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n n!}z^n \tag{1}
$$

if $p \le q$, $z \in \mathbb{C}$. If $p = q+1$ the above series only converges in the open unit disk and ${}_pF_q(z)$ is defined as analytic continuation of its sum for $z \in \mathbb{C}\setminus[1,\infty)$. Here $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the rising factorial (or Pochhammer's symbol) and $\mathbf{a} = (a_1,\ldots,a_p)$, $\mathbf{b} = (b_1,\ldots,b_q)$ are (generally complex) parameter vectors, such that $-b_j \notin \mathbb{N}_0$, $j = 1,\ldots,q$. This last restriction can be easily removed by dividing both sides of (1) by $\prod_{k=1}^{q}\Gamma(b_k)$. The resulting function (known as the regularized generalized hypergeometric function) is entire in $\mathbf{b}$. One useful tool in the study hypergeometric functions is their integral representations. Probably, the earliest such representation is given by Euler's integral

$$
{}_2F_1\,(\sigma, a; b; -z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1\frac{t^{a-1}(1-t)^{b-a-1}}{(1+zt)^{\sigma}}dt,
$$

that is finite for $z \in \mathbb{C}\setminus(-\infty,-1]$, $\Re(b-a) > 0$ and $\Re(a) > 0$. This formula can be interpreted as the generalized Stieltjes transform of the beta density $t^{a-1}(1-t)^{b-a-1}$. In a recent paper [28], Koornwinder generalized this formula and related it to fractional integration formulas and transmutation operators. See also references in [28] for the history of the subject. Similar formulas with the generalized Stieltjes transform replaced by the Laplace and cosine Fourier transforms are valid for ${}_1F_1$ and ${}_0F_1$, respectively. It seems surprising that for $p > 1$ the generalized Stieltjes transform representation of ${}_{p+1}F_p$ (as well as the Laplace and cosine Fourier transform representations for ${}_pF_p$ and ${}_{p-1}F_p$) has been only derived in 1994 by Kiryakova in her book [26, Chapter 4] and the article [27] by the same author. Her method of proof involves consecutive fractional integrations and requires the restrictions $b_j > a_j > 0$ on parameters in (2). We rediscovered similar representation using a different method in [24] and utilized it to derive various inequalities and monotonicity results for ${}_{p+1}F_p$. Next, we relaxed the restrictions $b_j > a_j > 0$ by demonstrating in [20, Theorem 2] that, for an arbitrary complex σ, the representation

$$
{}_{p+1}F_p\left(\begin{matrix}\sigma,\mathbf{a}\\ \mathbf{b}\end{matrix}\middle| -z\right) = \int_0^1\frac{\rho(s)ds}{(1+sz)^{\sigma}} \tag{2}
$$

holds with a summable function ρ and $|\arg(1+z)| < \pi$ if and only if $\Re a_i > 0$ for $i = 1,\ldots,p$ and $\Re\psi_p > 0$, where $\psi_p := \sum_{j=1}^{p}(b_j - a_j)$. In the affirmative case

$$
\rho(s) = \frac{\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})}G^{p,0}_{p,p}\left(s\,\middle|\begin{matrix}\mathbf{b}-1\\ \mathbf{a}-1\end{matrix}\right), \tag{3}
$$

where $G^{p,0}_{p,p}$ is Meijer's G-function defined in (8). Further details about this function will be given in the subsequent section and can be found in [5, Section 12.3], [12, Section 5.3], [25, Chapter 1], [41, Section 8.2] and [3, Section 16.17]. In (3) we have used the abbreviated notation $\Gamma(\mathbf{a})$ to denote the product $\prod_{i=1}^{p}\Gamma(a_i)$. This convention will also be used in the sequel. The sums like $\mathbf{b}+\alpha$ for a scalar α and inequalities like $\mathbf{a} > 0$ will always be understood element-wise,

Please cite this article in press as: D.B. Karp, J.L. López, Representations of hypergeometric functions for arbitrary parameter values and their use, Journal of Approximation Theory (2017), http://dx.doi.org/10.1016/j.jat.2017.03.004