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Error bounds for multiquadrics without added constants

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Dedicated to the memory of Professor M J D Powell FRS (1936–2015)

Abstract

While it was noted by R. Hardy and proved in a famous paper by C. A. Micchelli that radial basis function interpolants $s(x) = \sum \lambda_j \phi(\|x - \mathbf{x}_j\|)$ exist uniquely for the multiquadric radial function $\phi(r) = \sqrt{r^2 + c^2}$ as soon as the (at least two) centres are pairwise distinct, the error bounds for this interpolation problem always demanded an added constant to s. By using Pontryagin native spaces, we obtain error bounds that no longer require this additional constant expression.

1. Introduction

Among the various approaches to the approximation of continuous multivariable functions, the approximands (the continuous functions to be approximated), by simpler approximants from linear spaces, the method of radial basis functions has obtained in the last two decades a popular and successful place (Cheney and Light, 1999 [12]). Its central idea is, given a function $f : \mathbb{R}^d \to \mathbb{R}$, that is at a minimum continuous, and so-called centres $\mathbf{x}_i \in \mathbf{X} \subset \mathbb{R}^d$ which serve to define the linear space of approximants – often at the same time to place positions where

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approximand and approximant have to meet by means of interpolation – we use

$$s(x) = \sum_{\mathbf{x}_i \in \mathbf{X}} \lambda_{\mathbf{x}_i} K(\mathbf{x}, \mathbf{x}_i), \qquad x \in \mathbb{R}^d,$$

for the approximation.

Here, in addition to the given centres, the kernel K is often defined as

$$K(\mathbf{x}, \mathbf{y}) = \phi(\|\mathbf{x} - \mathbf{y}\|)$$

with the help of a univariate, continuous function ϕ; the latter is the radial basis function which is made multi-variate and radially symmetric by composition with the d-variate Euclidean norm $\|\cdot\|$. A great number of radial basis functions have turned out to be mathematically attractive and useful in several applications. A useful summary of theory and applications of radial basis functions is given for example in the book (Buhmann, 2003 [9]). For computational issues, see for instance (Beatson and Greengard, 1997 [4]).

Among them are the thin-plate splines $\phi(r) = r^2 \log r$ of (Duchon, 1979 [15]), the shifted version thereof – to avoid the removable singularity at the origin – $\phi(r) = (r^2 + c^2) \log(r^2 + c^2)$ (Dyn, 1987 [16]), a variety of (non-even) positive powers of r, and the famous multiquadrics and inverse multiquadrics, $\phi(r) = \sqrt{r^2 + c^2}$ and its reciprocal, respectively.

One reason why these radial basis functions are so attractive is that they give rise to the aforementioned interpolation problems

$$s(\mathbf{x}_j) = f(\mathbf{x}_j), \qquad \mathbf{x}_j \in \mathbf{X},$$

which turns out to be regular or even positive definite (that is, with a positive definite interpolation matrix) for several choices of radial basis functions. Among those that provide even positive definite interpolation linear systems are the inverse multiquadrics and the Gauß- and Poisson kernels ($\phi(r) = \exp(-c^2 r^2)$ and $\phi(r) = \exp(-c^2 r)$, respectively).

In all these cases, c is a positive constant. As is well known, the positive definiteness of the interpolation matrix

$$A = \{\phi(\|\mathbf{x}_i - \mathbf{x}_j\|)\}_{\mathbf{x}_i, \mathbf{x}_j \in \mathbf{X}}$$

is related to the complete monotonicity of the functions $g(t) = \phi(\sqrt{t})$ (Micchelli, 1986 [19], Powell, 1987 [20], Schoenberg, 1938 [21]).

However, we know that not all radial basis functions, and not even all of them that are mentioned in this introduction, have this property. Many of them are only conditionally positive definite because their radial part composed with the square root is not completely monotonic, but only a derivative thereof is up to a sign change and not constant. Examples are multiquadrics and (shifted) thin-plate splines which are (subject to a straightforward sign change) conditionally positive definite of order one and two, respectively. This means normally that we have to add a polynomial of that order (its degree is one less) to the approximant. The additional degrees of freedom are taken up by adding side conditions on the coefficients of this type:

$$\sum_{\mathbf{x}_i \in \mathbf{X}} \lambda_{\mathbf{x}_i} p(\mathbf{x}_i) = 0, \qquad \forall\, p \in \Pi_k^d,$$

the notation Π_k^d being for the linear space of polynomials of order at most k in d unknowns. Due to this, one would normally add a constant (polynomial) to the multiquadrics approximant in d dimensions. However, it was noted by Micchelli (1986) [19] that this is actually not needed to guarantee the regularity of the interpolation matrix A. Therefore the question arises about the convergence estimates of such approximants without the polynomial added.