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Szegő–Widom asymptotics of Chebyshev polynomials on circular arcs

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Abstract

Thiran and Detaille give an explicit formula for the asymptotics of the sup-norm of the Chebyshev polynomials on a circular arc. We give the so-called Szegő–Widom asymptotics for this domain, i.e., explicit expressions for the asymptotics of the corresponding extremal polynomials. Moreover, we solve a similar problem with respect to the upper envelope of a family of polynomials uniformly bounded on this arc. That is, we give explicit formulas for the asymptotics of the error of approximation as well as of the extremal functions. Our computations show that in the proper normalization the limit of the upper envelope represents the diagonal of a reproducing kernel of a certain Hilbert space of analytic functions. Due to Garabedian, the analytic capacity in an arbitrary domain is the diagonal of the corresponding Szegő kernel. We do not know any result of this kind with respect to upper envelopes of polynomials. If this is a general fact or a specific property of the given domain, we rise as an open question.

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1. Introduction

Getting explicit asymptotics is a fundamental problem in constructive approximation theory. The problem can be related to both, approximation error and the function of best approximation.

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Usually, the second problem is essentially harder. For instance, it took almost 100 years that Lubinsky [10] characterized the extremal function in the famous Bernstein problem on the best approximation in $[-1, 1]$ of $|x|^\alpha$ by polynomials.

In the following we will use some basic concepts of potential theory, see e.g. [8,13]. The nth Chebyshev polynomial, T_n, associated to a compact set $E \subset \mathbb{C}$ is the unique polynomial of degree n that minimizes the sup-norm

$$\|T_n\|_E = \sup_{z \in E} |T_n(z)|.$$

If E is a finite union of Jordan regions, extending a result of Akhiezer [1,2] and Faber [6], Widom described the asymptotics of the Chebyshev polynomials completely (i.e., asymptotics of the extremal value and function) in his seminal work [16]. In particular, for the extremal value he showed that $\|T_n\|_E \sim \mu_n \mathrm{Cap}(E)^n$, where $\mathrm{Cap}(E)^n$ denotes the logarithmic capacity and μ_n is a rather explicit factor. If E also contains some arc-components he was not able to complete the theory, but showed that if E is a finite union of intervals

$$\|T_n\|_E \sim 2\mu_n \mathrm{Cap}(E)^n. \tag{1.1}$$

Recently Christiansen, Simon and Zinchenko [5] (to appear in Invent. math.) also found the asymptotics of the extremal functions T_n in this case. Moreover, for infinite gap sets $E \subset \mathbb{R}$ of Parreau–Widom type, they found an upper bound for the ratio $\|T_n\|_E / \mathrm{Cap}(E)^n$.

Motivated by the real case, Widom conjectured that the factor 2 in (1.1) should always appear, whenever an arc-component is present in E. This was proved to be wrong by Thiran and Detaille [14], who showed that the norm of the Chebyshev polynomials related to the circular arcs

$$A_\alpha = \{u \in \mathbb{C} : \ |u| = 1, \ -\alpha \le \arg u \le \alpha\}, \quad 0 < \alpha < \pi$$

behaves like

$$\|T_n\|_{A_\alpha} \sim 2\cos(\alpha/4)^2 \mathrm{Cap}(A_\alpha)^n. \tag{1.2}$$

Note that for simply connected domains $\mu_n = 1$. A more detailed study on the asymptotic behavior of $\|T_n\|_E$, where E has arc-components, was later given by Totik and Yuditskii [15]. In the present paper we study the Chebyshev polynomials associated to A_α. Our approach is completely different from [14] and allows us to find:

(i) explicit asymptotics of the Chebyshev polynomials,
(ii) explicit asymptotics of the *upper envelope* of the family $\mathcal{P}_{n,\alpha}$ of polynomials of degree at most n which are bounded by one in modulus on A_α; cf. (2.1).

To be more precise, let $g_\Omega(z, z_0)$ denote the Green's function of the point z_0 and the domain Ω. Writing $i * g_\Omega(z, z_0)$ for the harmonic conjugate of $g_\Omega(z, z_0)$, we define the complex Green's function of the domain by

$$b_\Omega(z, z_0) = e^{-(g_\Omega(z,z_0) + i * g_\Omega(z,z_0))};$$

see e.g. [16]. Instead of the Chebyshev polynomial, let us consider the normalized polynomial $P_{n,\infty} = T_n / \|T_n\|_{A_\alpha}$, i.e., the polynomial in $\mathcal{P}_{n,\alpha}$ that has maximal leading coefficient. Set $\Omega_\alpha = \overline{\mathbb{C}} \setminus A_\alpha$. Due to Montel's theorem, at least by passing to subsequences, the family $b_{\Omega_\alpha}(u, \infty)^n P_{n,\infty}(u)$ has a limit as $n \to \infty$. We present the limit function explicitly by means of a conformal mapping. Following the notion introduced in [5], we say that Ω_α has Szegő–Widom asymptotics.