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Journal of Approximation Theory 216 (2017) 86–116

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

Full length article

Needlet approximation for isotropic random fields on the sphere[☆]

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Received 11 January 2016; received in revised form 10 December 2016; accepted 3 January 2017
Available online 16 January 2017

Communicated by Gitta Kutyniok

Abstract

In this paper we establish a multiscale approximation for random fields on the sphere using spherical needlets—a class of spherical wavelets. We prove that the semidiscrete needlet decomposition converges in mean and pointwise senses for weakly isotropic random fields on $\mathbb{S}^d$, $d \geq 2$. For numerical implementation, we construct a fully discrete needlet approximation of a smooth 2-weakly isotropic random field on $\mathbb{S}^d$ and prove that the approximation error for fully discrete needlets has the same convergence order as that for semidiscrete needlets. Numerical examples are carried out for fully discrete needlet approximations of Gaussian random fields and compared to a discrete version of the truncated Fourier expansion.

MSC: 60G60; 42C40; 41A25; 65D32; 60G15; 33G55

Keywords: Isotropic random fields; Sphere; Needlets; Gaussian; Multiscale

☆ This research was supported under the Australian Research Council's *Discovery Projects* DP120101816 and DP160101366. The first author was supported under the University International Postgraduate Award (UIPA) of UNSW Australia.

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http://dx.doi.org/10.1016/j.jat.2017.01.001

1. Introduction

Isotropic random fields on the sphere have application in environmental models and astrophysics, see [9,11,17,30,35,36]. It is well-known that a 2-weakly isotropic random field on the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$, $d \geq 2$, has a Karhunen–Loève (K–L) expansion in terms of spherical harmonics, see e.g. [18,21]. In this paper we establish a multiscale approximation for spherical random fields using spherical needlets [25,26]. We prove that the semidiscrete needlet approximation converges in both mean and pointwise senses for a weakly isotropic random field on $\mathbb{S}^d$. We establish a fully discrete needlet approximation by discretizing the needlet coefficient integrals by means of suitable quadrature (numerical integration) rules on $\mathbb{S}^d$. We prove that when the field is sufficiently smooth and the quadrature rule has sufficiently high polynomial degree of precision the error of the approximation by fully discrete needlets has the same convergence order as that for semidiscrete needlets. An algorithm for fully discrete needlet approximations is given, and numerical examples using Gaussian random fields are provided.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability measure space. For $1 \leq p \leq \infty$, let $\mathbb{L}_p(\Omega, \mathrm{P})$ be the $\mathbb{L}_p$-space on Ω with respect to the probability measure P, endowed with the norm $\|\cdot\|_{\mathbb{L}_p(\Omega)}$. Let $\mathbb{E}[X]$ denote the *expected value* of a random variable X on $(\Omega, \mathcal{F}, \mathrm{P})$.

Random fields. A *real-valued random field* on the sphere $\mathbb{S}^d$ is a function $T : \Omega \times \mathbb{S}^d \to \mathbb{R}$. We write $T(\omega, \mathbf{x})$ by $T(\mathbf{x})$ or $T(\omega)$ for brevity if no confusion arises. We say T is 2-weakly isotropic if its expected value and covariance are rotationally invariant.

Spherical needlets [25,26] ψ_{jk} are localized polynomials on the sphere associated with a *quadrature rule* and a *filter*; see (2.10)–(2.12) below. The level-j needlet ψ_{jk}, $k = 1, \ldots, N_j$, is a spherical polynomial of degree $2^j - 1$, associated with a positive weight quadrature rule with degree of precision $2^{j+1} - 1$.

Main results

Needlets form a multiscale tight frame for square-integrable functions on $\mathbb{S}^d$, see [25,26]. The resulting tight frame has good approximation performance for random fields on the sphere.

Needlet decomposition for random fields. For a random field T on $\mathbb{S}^d$, the *(semidiscrete) needlet approximation* of order J for $J \in \mathbb{N}_0$ is defined as

$$V_J^{\mathrm{need}}(T; \omega, \mathbf{x}) := \sum_{j=0}^{J} \sum_{k=1}^{N_j} (T(\omega), \psi_{jk})_{\mathbb{L}_2(\mathbb{S}^d)} \psi_{jk}(\mathbf{x}), \quad \omega \in \Omega, \ \mathbf{x} \in \mathbb{S}^d. \tag{1.1}$$

Let $\lceil \cdot \rceil$ be the ceiling function. Given $1 \leq p < \infty$ and $d \geq 2$, the needlet approximation given by (1.1) of a $\lceil p \rceil$-weakly isotropic random field T on $\mathbb{S}^d$ converges to T in $\mathbb{L}_p(\Omega \times \mathbb{S}^d)$. (See Theorem 3.4, and see Section 2 for the definition of n-weakly isotropic, $n \in \mathbb{N}$.)

In Theorem 3.9, we also prove that when p is a positive integer the needlet approximation on the set of p-weakly isotropic random fields is bounded in the $\mathbb{L}_p(\Omega \times \mathbb{S}^d)$ sense.

Let $d \geq 2$. Let T be a 2-weakly isotropic random field satisfying $\sum_{\ell=1}^{\infty} A_\ell \ell^{2s+d-1} < \infty$. Here A_ℓ is the angular power spectrum of T, see Section 4.1. Then $T(\omega)$ is in the Sobolev space $\mathbb{W}_2^s(\mathbb{S}^d)$ P-almost surely (or P-a.s.). (This fact was proved in [18, Section 4] and [3]; we restate this result in Corollary 4.4.)

Semidiscrete needlet approximation. In Theorems 4.6 and 4.7, we prove that the semidiscrete needlet approximation $V_J^{\mathrm{need}}(T)$ has the following mean and pointwise approximation errors. Let $s > 0$. Then

$$\mathbb{E}\left[\|T - V_J^{\mathrm{need}}(T)\|_{\mathbb{L}_2(\mathbb{S}^d)}^2\right] \leq c 2^{-2Js} \mathbb{E}\left[\|T\|_{\mathbb{W}_2^s(\mathbb{S}^d)}^2\right]. \tag{1.2}$$