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Full length article

Improvement of the lower bound of the PCM quantization error for vectors in $\mathbb{R}^2$

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Abstract

Using an asymptotic estimate of the Bessel functions, we investigate the performance of the PCM quantization for vectors in $\mathbb{R}^2$ and improve the previous results of Wang and Xu (2012). In particular, we present a lower bound of the PCM quantization error under some mild condition and show that PCM quantization can take the advantage of the redundancy of frames for infinitely many $x \in \mathbb{R}^2$ with different magnitudes.

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1. Introduction

In signal processing, one needs to find a digital representation for a given signal that is suitable for storage, transmission, and recovery. We assume that the signal $x \in \mathbb{R}^d$. Suppose

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that $\mathcal{F} := \{e_j\}_{j=1}^N \subset \mathbb{R}^d$. We say that $\mathcal{F}$ is a *unit-norm tight frame* of $\mathbb{R}^d$ if

$$x = \frac{d}{N}\sum_{j=1}^{N}\langle x, e_j\rangle e_j$$

holds for all $x \in \mathbb{R}^d$ and $\|e_j\|_2 = 1$ holds for all $1 \leq j \leq N$. In the digital domain the coefficients $x_j = \langle x, e_j\rangle$ must be mapped to a discrete set of values $\mathcal{A}$ which is called the quantization alphabet. A natural way to achieve such a mapping is the Pulse Code Modulation (PCM) quantization scheme. For the PCM scheme, the mapping is done by the function

$$Q_\delta(t) := \operatorname*{argmin}_{r\in\mathcal{A}} |t - r|.$$

If $\mathcal{A} = \delta\mathbb{Z}$ with $\delta > 0$, we have

$$Q_\delta(t) = \delta\left\lfloor \frac{t}{\delta} + \frac{1}{2}\right\rfloor.$$

Thus in practical applications we in fact have only a quantized representation

$$q_j := Q_\delta(\langle x, e_j\rangle), \quad j = 1, \ldots, N$$

for each $x \in \mathbb{R}^d$. The linear reconstruction is

$$\tilde{x}_{\mathcal{F}} = \frac{d}{N}\sum_{j=1}^{N} Q_\delta(\langle x, e_j\rangle)e_j.$$

One investigates the performance of PCM scheme by considering

$$E_\delta(x, \mathcal{F}) := \|x - \tilde{x}_{\mathcal{F}}\|,$$

where $\|\cdot\|$ is ℓ_2 norm. To simplify the investigation of $E_\delta(x, \mathcal{F})$, one employs the *White Noise Hypothesis* (WNH) in this area (see [1,6,4,3,2,5,7]), which asserts that the quantization error sequence $\{x_j - q_j\}_{j=1}^N$ can be modeled as an independent sequence of i.i.d. random variables that are uniformly distributed on the interval $(-\delta/2, \delta/2)$. Under the WNH, one can obtain the mean square error

$$MSE = \mathbb{E}(\|x - \tilde{x}_{\mathcal{F}}\|^2) = \frac{d^2\delta^2}{12N}.$$

The result implies that the MSE of $E_\delta(x, \mathcal{F})$ tends to 0 with N tending to infinity. However, as pointed out in [5,7], the WNH only asymptotically holds for fine quantizations (i.e. as δ tends to 0) under rather general conditions. So, for a fixed x, one is interested in estimating $E_\delta(x, \mathcal{F})$ without WNH. The result in [10] gives a solution for the case $d = 2$ which shows that for some $x \in \mathbb{R}^2$ the quantization error $E_\delta(x, \mathcal{F})$ does not diminish to 0 with N tending to infinity.

Theorem 1.1 (*[10]*). *Set* $x_\psi := r[\cos\psi, \sin\psi]^\top$ *where* $r > 0$. *Set* $R := r/\delta$, $\varepsilon := R - \lfloor R\rfloor$, *and*

$$\mathbb{E}_\delta(r, \mathcal{F}) := \left(\int_0^{2\pi} |E_\delta(x_\psi, \mathcal{F})|^2 d\psi\right)^{1/2}.$$