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# Limits for circular Jacobi beta-ensembles

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**Abstract**

Bourgade, Nikeghbali and Rouault recently proposed a matrix model for the circular Jacobi $\beta$-ensemble. This is a generalization of the Dyson circular $\beta$-ensemble but equipped with an additional parameter $b$ controlling the order of a spectrum singularity. We calculate the scaling limits for expected products of characteristic polynomials of circular Jacobi $\beta$-ensembles. For a fixed constant $b$, the resulting limit near the spectrum singularity is proven to be a new multivariate function. When $b = \beta N d/2$, the scaling limits in the bulk and at the soft edge agree with those of the Hermite (Gaussian), Laguerre (chiral) and Jacobi $\beta$-ensembles proved in Desrosiers and Liu (2014). As corollaries, for even $\beta$ the scaling limits of point correlation functions for the ensemble are given. Besides, a transition from the spectrum singularity to the soft edge limit is observed as $b$ goes to infinity. The positivity of two special multivariate hypergeometric functions, which appear as one factor of the joint eigenvalue densities for Jacobi/Wishart $\beta$-ensembles with general covariance and Gaussian $\beta$-ensembles with source, will also be shown.



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## 1. Introduction

### 1.1. Circular Jacobi ensembles

The circular unitary ensemble in random matrix theory refers to the unitary group $U(N)$ with its invariant Haar measure. The correlation functions and moments of characteristic polynomials for eigenvalues of these random unitary matrices are predicted to be somehow related with statistical properties of the Riemann zeta function in number theory [42]. Together with the circular orthogonal and symplectic ensembles (see [31, Chap. 2] for their definition and relationship to compact symmetric spaces) they specify Dyson's three-fold way for circular ensembles. Dyson also observed that the induced eigenvalue densities correspond to the Gibbs measure of Coulomb log-gases on the circle at three different inverse temperatures $\beta = 1, 2, 4$. For general $\beta > 0$ these are called $\beta$-ensembles in the literature.

Inspired by Dumitriu–Edelman tridiagonal matrix models for Gaussian and Laguerre $\beta$-ensembles [24], Killip and Nenciu constructed analog matrix models for Dyson circular $\beta$-ensembles and also for Jacobi $\beta$-ensembles [45]. Combining circular and Jacobi ensembles, a more general ensemble was defined as the circular Jacobi ensemble in [31,52]. Explicitly, the circular Jacobi ensemble refers to the probability density function on $[0, 2\pi)^N$

$$P_{b,N}(\theta_1, \ldots, \theta_N) = \frac{1}{M_N(\bar{b}, b; \beta/2)} \prod_{j=1}^{N} \left( e^{i\frac{\bar{b}-b}{2}(\theta_j - \pi)} |1 - e^{i\theta_j}|^{\bar{b}+b} \right) |\Delta_N(e^{i\theta})|^{\beta} \tag{1.1}$$

with $\mathrm{Re}\{b\} > -1/2$, see [31, Chap. 3] for detailed discussion and [14] for the matrix model realization. Here the constant $M_N(a', b'; \alpha)$ equals to (see, e.g., [31, Chap. 4])

$$M_N(a', b'; \alpha) = (2\pi)^N \prod_{j=0}^{N-1} \frac{\Gamma(1 + \alpha + j\alpha)\Gamma(1 + a' + b' + j\alpha)}{\Gamma(1 + \alpha)\Gamma(1 + a' + j\alpha)\Gamma(1 + b' + j\alpha)}, \tag{1.2}$$

and $\Delta_N$ is the Vandermonde product

$$\Delta_N(e^{i\theta}) = \prod_{1 \le j < k \le N} (e^{i\theta_k} - e^{i\theta_j}).$$

For $\beta = 2$, such measures were first studied by Hua [36] and later by Nagao [52] motivated by random matrix theory. For $\beta = 1, 2, 4$, the point correlation functions and their scaling limits were investigated by Forrester and Nagao [33]. Note that for $b = 0$ (1.1) reduces to the density for the circular ensemble and for $b \in \frac{\beta}{2}\mathbb{N}$ it coincides with the circular ensemble given that there is an eigenvalue at $\theta = 0$. For general $b$ it is said to describe a spectrum singularity.

Applying an appropriate change of variables we can derive some classical random matrix ensembles from (1.1). For instance, with

$$e^{i\theta_j} = \frac{1 + i\lambda_j}{1 - i\lambda_j}, \quad j = 1, 2, \ldots, N,$$

(1.1) reduces to the Cauchy ensemble, which shows that both ensembles are in fact equivalent, see Chapter 3.9 of [31]. Another example is that the Laguerre ensemble can also be treated as a limit transition from (1.1). The latter fact is a remark of E.M. Rains (AIM workshop 2009) communicated to the author by P.J. Forrester, which can be formally inferred as follows. Let