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Sharp constants in V. A. Markov–Bernstein type inequalities of different metrics

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Abstract

We study relations between sharp constants in the V. A. Markov–Bernstein inequalities of different L_r-metrics for algebraic polynomials on an interval and for entire functions on the real line or half-line. In a number of cases, we prove that the sharp constant in the inequality for entire functions of exponential type or semitype is the limit of sharp constants in the corresponding inequalities for algebraic polynomials of degree n as $n \to \infty$.

MSC: 41A17; 41A10

Keywords: V. A. Markov–Bernstein inequality of different metrics; Algebraic polynomials; Trigonometric polynomials; Entire functions of exponential type; Entire functions of exponential semitype

1. Introduction

In this paper we discuss limit relations between the sharp constants in the univariate V. A. Markov–Bernstein type inequalities of different L_r-metrics for algebraic polynomials and entire functions of exponential type and semitype.

Notation and preliminaries. Let $\mathcal{P}_n$ and $\mathcal{T}_n$ be the sets of all algebraic and trigonometric polynomials of degree at most n with complex coefficients, respectively. The set of all

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complex-valued entire functions of exponential type $\sigma > 0$ is denoted by B_σ and the set of all complex-valued entire functions of order 1/2 and type $\sigma > 0$ (i.e. entire functions of exponential semitype σ; cf. Bernstein's terminology in [6]) is denoted by $B_{1/2,\sigma}$. Let $L_r(E)$ be the space of all measurable complex-valued functions F on a measurable subset E of the real line $\mathbb{R}$ with the finite quasinorm

$$\|F\|_{L_r(E)} := \begin{cases} \left(\int_E |F(x)|^r dx\right)^{1/r}, & 0 < r < \infty, \\ \operatorname*{ess\,sup}_{x\in E} |F(x)|, & r = \infty. \end{cases}$$

Next, we define sharp constants in V. A. Markov–Bernstein type inequalities of different metrics for algebraic and trigonometric polynomials and entire functions of exponential type and semitype. Let

$$M_{p,n}^{(s)} := [(b-a)/(2n)]^{s+1/p} \sup_{P\in\mathcal{P}_n\setminus\{0\}} \left(\left|P^{(s)}((a+b)/2)\right| / \|P\|_{L_p([a,b])}\right), \tag{1.1}$$

$$N_{p,q,n}^{(s)} := [(b-a)/n^2]^{s+1/p-1/q} \sup_{P\in\mathcal{P}_n\setminus\{0\}} \left(\|P^{(s)}\|_{L_q([a,b])} / \|P\|_{L_p([a,b])}\right), \tag{1.2}$$

$$A_{p,n}^{(s)} := n^{-s-1/p} \sup_{T\in\mathcal{T}_n\setminus\{0\}} \left(\|T^{(s)}\|_{L_\infty([-\pi,\pi])} / \|T\|_{L_p([-\pi,\pi])}\right), \tag{1.3}$$

$$D_p^{(s)} := \sigma^{-s-1/p} \sup_{f\in(B_\sigma\cap L_p(\mathbb{R}))\setminus\{0\}} \left(\|f^{(s)}\|_{L_\infty(\mathbb{R})} / \|f\|_{L_p(\mathbb{R})}\right), \tag{1.4}$$

$$E_{p,q}^{(s)} := (2/\sigma)^{2(s+1/p-1/q)} \sup_{h\in(B_{1/2,\sigma}\cap L_p([0,\infty)))\setminus\{0\}} \left(\|h^{(s)}\|_{L_q([0,\infty))} / \|h\|_{L_p([0,\infty))}\right). \tag{1.5}$$

Here, $n \in \mathbb{N} := \{1, 2, \ldots\}$, $\sigma \in (0, \infty)$, $s \in \mathbb{Z}_+ := \{0, 1, 2, \ldots\}$, and $0 < p \le q \le \infty$. If the suprema in (1.1) through (1.5) are taken over all real-valued functions on $\mathbb{R}$ from $\mathcal{P}_n \setminus \{0\}$, $\mathcal{T}_n \setminus \{0\}$, $B_\sigma \setminus \{0\}$, and $B_{1/2,\sigma} \setminus \{0\}$, the corresponding real constants are denoted by $M_{p,n,\mathbb{R}}^{(s)}$, $N_{p,q,n,\mathbb{R}}^{(s)}$, $A_{p,n,\mathbb{R}}^{(s)}$, $D_{p,\mathbb{R}}^{(s)}$, and $E_{p,q,\mathbb{R}}^{(s)}$, respectively.

Note that the constant $E_{p,q}^{(s)}$ does not depend on σ since $h \in B_{1/2,\sigma}$ if and only if $g(x) := h(x/\sigma^2) \in B_{1/2,1}$ and

$$\begin{aligned} E_{p,q}^{(s)} &= 2^{2s}(2/\sigma)^{2(1/p-1/q)} \sup_{g\in(B_{1/2,\sigma}\cap L_p([0,\infty)))\setminus\{0\}} \left(\|g^{(s)}(\sigma^2\cdot)\|_{L_q([0,\infty))} / \|g(\sigma^2\cdot)\|_{L_p([0,\infty))}\right) \\ &= 2^{2(s+1/p-1/q)} \sup_{g\in(B_{1/2,1}\cap L_p([0,\infty)))\setminus\{0\}} \left(\|g^{(s)}\|_{L_q([0,\infty))} / \|g\|_{L_p([0,\infty))}\right). \end{aligned}$$

The constant $D_p^{(s)}$ is independent of σ as well (see [12]). In addition, it is easy to verify by the corresponding linear substitutions that the constants $M_{p,n}^{(s)}$ and $N_{p,q,n}^{(s)}$ do not depend on a and b. In the sequel, we assume that $\sigma = 1$ in (1.4) and (1.5). In addition, we will often use the intervals $[a, b] = [-1, 1]$ in (1.1) and $[a, b] = [0, 1]$ in (1.2). The real constants have the same invariance properties and we will use the same intervals for these constants as well.

Note that

$$M_{p,n}^{(s)} := \frac{1}{n^{s+1/p}} \sup_{P\in\mathcal{P}_n\setminus\{0\}} \frac{\left|P^{(s)}(0)\right|}{\|P\|_{L_p([-1,1])}}$$