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Notes

Shifts of a measurable function and criterion of p-integrability

Boris S. Mityagin

231 West 18th Avenue, The Ohio State University, Columbus, OH 43210, United States

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Abstract

It is shown that two conditions $f(a+\cdot)-f(\cdot)\in L^p(\mathbb{R})$, and $(\sin b\cdot)f(\cdot)\in L^p(\mathbb{R})$ guarantee $f\in L^p(\mathbb{R})$, $1\le p<\infty$, if and only if ab is not in $(\pi\mathbb{Z})$.

MSC: 46E30

Keywords: L^p spaces; Banach spaces of measurable functions

Когда б вы знали, из какого сора
Растут стихи, не ведая стыда,
Как желтый одуванчик у забора,
Как лопухи и лебеда.

А. Ахматова, ‘Тайны ремесла’[1]

1.

Let a measurable function f on $\mathbb{R}=(-\infty,\infty)$ have properties

$$\forall t\in\mathbb{R},\qquad f(t+\cdot)-f(\cdot)\in L^2(\mathbb{R}),\tag{1a}$$

E-mail address: mityagin.1@osu.edu.

[1] English translation is given in Appendix.

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and

$$\forall s \in \mathbb{R}, \qquad \sin(s\cdot) f(\cdot) \in L^2(\mathbb{R}). \tag{1b}$$

If a Fourier transform $\widetilde{f}$ is reasonably defined then (1b) is equivalent to (1a) for $\widetilde{f}$.

Claim 1. *Under conditions (1a), (1b) we have* $f \in L^2(\mathbb{R})$.

Recently, A. M. Vershik brought attention of the 25th St. Petersburg Summer Meeting in Mathematical Analysis, June 25–30, 2016, to Claim 1. He recalled that the known proof "was done in terms of representation theory (of Heisenberg group) many years ago" but noted that "the simple proof still does not exist" and after many years it is important "to give a simple and direct proof".[2] A stronger form of Claim 1 and its elementary proof was given just during the Meeting's session of A. Vershik's talk on June 30. It is presented in Section 2. If the reader wants a proof only of Claim 1, there is no need to go beyond Section 2.

2.

Claim 2. *Let f be a measurable function on $\mathbb{R}$, and the following two conditions hold:*

$$\Delta(x) = f\left(x + \frac{\pi}{2}\right) - f(x) \in L^2(\mathbb{R}). \tag{2a}$$

$$f(x) \cdot \sin(x) \in L^2(\mathbb{R}) \tag{2b}$$

Then $f \in L^2(\mathbb{R})$.

Proof. Put $E = \{x : |x - k\pi| \le 10^{-6}$ for some $k \in \mathbb{Z}\}$. Then $|\frac{1}{\sin x}| \le \frac{1}{\sin\delta} \le 10^7$ on $E^{\complement} = \mathbb{R} \setminus E, \quad \delta = 10^{-6}$, so

$$f|\,E^{\complement} = (\sin x \cdot f(x)) \cdot \frac{1}{\sin x} \in L^2(E^{\complement}). \tag{3}$$

With $2\delta < \frac{\pi}{2}$ we have $E + \frac{\pi}{2} \subset E^{\complement}$ and $f(x) = f(x + \frac{\pi}{2}) - \Delta(x)$ for $x \in E$; therefore $\|f|E\| \le \|f|E^{\complement}\| + \|\Delta\| < \infty$, and together with (3) and (2a) we have $f \in L^2(\mathbb{R})$. □

3.

Section 2 is an almost stenographic recording of what I have said at the Meeting's June 30 session. Now we will talk about a more general setting (sorry, some repetition is unavoidable) and get negative results (Proposition 5 and Example 8) as well. Of course, L^2-norm is not special in our analysis in Section 2. Instead of L^2 we can talk about any Banach space X of measurable functions on $\mathbb{R}$ with two properties:

$$g \in X \Rightarrow g(\cdot + t) \in X, \quad t \in \mathbb{R} \tag{4a}$$

$$g \in X \Rightarrow g \cdot h \in X, \quad \forall h \in L^\infty(\mathbb{R}). \tag{4b}$$

Moreover, we do not need global conditions (1a), (1b); just a pair $(t; s) = \left(\frac{\pi}{2}; 1\right)$ with (2a), (2b) holding was good enough for the proof in Section 2. More general than Claim 2 is true:

[2] The presentation [2] gives a more extended motivation and links to the uncertainty principle although no reference to a published source is given.