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Minimal cubature rules and polynomial interpolation in two variables II

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Abstract

As a complement to Xu (2012), minimal cubature rules of degree $4m + 1$ for the weight functions

$$\mathcal{W}_{\alpha,\beta,\pm\frac{1}{2}}(x, y) = |x + y|^{2\alpha+1}|x - y|^{2\beta+1}((1 - x^2)(1 - y^2))^{\pm\frac{1}{2}}$$

on $[-1, 1]^2$ are shown to exist and near minimal cubature rules of the same degree with one node more than minimal are constructed explicitly. The Lagrange interpolation polynomials on the nodes of the near minimal cubature rules are also studied.

MSC: 41A05; 65D05; 65D32

Keywords: Cubature; Minimal cubature; Polynomial interpolation; Two variables; Lebesgue constant

1. Introduction

Let W be a non-negative weight function on a domain $\Omega \subset \mathbb{R}^2$. A cubature formula of degree s for the integral with respect to W is a finite sum satisfying

$$\int_{\Omega} f(x, y)W(x, y)dxdy = \sum_{k=1}^{N} \lambda_k f(x_k, y_k), \quad \forall f \in \Pi_s^2, \tag{1.1}$$

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where Π_s^2 denotes the space of polynomials of degree at most s in two variables, and there exists at least one function $f^* \in \Pi_{s+1}^2$ for which the identity fails to hold. All cubature rules appeared in this paper are positive in the sense that all λ_k are positive. For a fixed n, a minimal cubature rule of degree $2n+1$ has the smallest number, N, of nodes among all cubature rules of the same degree. They are of interest in several aspects and provide important tools for various problems in approximation and numerical computation.

Minimal cubature rules are known explicitly, with their nodes and weights given by closed formulas, only in a few cases. One of them is for the family of integrals with respect to the weight functions

$$\mathcal{W}_{\alpha,\beta,\pm\frac{1}{2}}(x, y) := |x + y|^{2\alpha+1}|x - y|^{2\beta+1}(1 - x^2)^{\pm\frac{1}{2}}(1 - y^2)^{\pm\frac{1}{2}} \tag{1.2}$$

with $\alpha, \beta > -1$ on the square $[-1, 1]^2$, for which the minimal cubature rules of degree $2n-1$ are explicitly constructed in [16] when $n = 2m$. This includes the classical result of the product Chebyshev weight function (when $\alpha = \beta = 1/2$) studied in [9] as a special case. The cubature rules in [16] are closely tied to Gaussian cubature rules on a domain bounded by two lines and a parabola. The nodes of these minimal cubature rules are common zeros of certain orthogonal polynomials with respect to $\mathcal{W}_{\alpha,\beta,\pm\frac{1}{2}}$, and, for each fixed n, there is a unique Lagrange interpolation polynomial based on the nodes of the minimal cubature rule. The case $n = 2m+1$ was left open in [16] because the ideal of the polynomials that vanish on its nodes has a more complicated structure that requires further study to understand.

The purpose of the present paper is to show how the case $n = 2m + 1$ can be resolved. We obtained two families of cubature rules in this case. The first family consists of minimal cubature rules, whose coefficients, however, are not explicitly given, whereas the second family consists of cubature rules whose number of nodes is 1 more than the theoretical lower bound, but it can be determined explicitly. The nodes of these cubature rules are common zeros of certain orthogonal polynomials of degree n and, in the case of the second family, one quasi-orthogonal polynomial of degree $n + 1$ that does not belong to the ideal generated by those orthogonal polynomials of degree n. The second family of cubature rules are explicitly constructed because they are related to the product Gauss–Radau cubature rules with respect to the product Jacobi weights. For all practical considerations, the second family is better and their study resembles the case of $n = 2m$ in [16]. In addition, we will also give explicit formulas of the Lagrange interpolation polynomials based on the nodes of the near minimal cubature rules. These formulas allow us to determine the order of the Lebesgue constants of the interpolation operators.

We regard this paper as a complement of [16] and will refer to the background materials and, in some cases, even quote formulas there. However, we have tried to make the paper self-contained, so that it can be read independently. The paper is organized as follows. In the next section we state background materials, highlight those not covered in [16]. The cubature formulas are studied in Section 3 and the Lagrange interpolation polynomials based on the nodes of the cubature rules are discussed in Section 4.

2. Preliminary and background

Besides the section on preliminary and background in [16], we need background on near minimal cubature rules and orthogonal polynomials of odd degrees with respect to the weight function $\mathcal{W}_{\alpha,\beta,\pm\frac{1}{2}}$.