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Average case tractability of a multivariate approximation problem☆,☆☆

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ABSTRACT

Many authors have studied exponentially-convergent tractability (EC-tractability) in the worst case setting. Here, we study EC-tractability in the average case setting. Our problem is multivariate approximation over the space of continuous real functions equipped with a zero-mean Gaussian measure whose covariance kernel is given as a Korobov kernel. We obtain necessary and sufficient conditions for certain kinds of tractability, including EC-tractability.

1. Introduction

Suppose that we are given a sequence of problems defined over $[0, 1]^d$ for any positive integer d. Roughly speaking, a problem is said to be *intractable* if the minimal cost of computing an ε-approximation to the true solution depends exponentially on either ε^{-1} or in d, and is said to be *tractable* otherwise. This idea was first introduced by H. Woźniakowski [16] in 1994. Since that time, tractability study has become one of the busiest areas of research in information-based complexity. Many kinds of tractability have been introduced since that time; the list includes such as strong polynomial tractability, polynomial tractability, quasi-polynomial tractability, weak tractability, uniform weak tractability and (s, t)-weak tractability. The inclusion relations among these notions can be found in Figure 1 in [15].

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Most work in tractability has dealt with problems defined over classes of functions with finite smoothness. For such problem classes, the corresponding minimal error sequence often converges polynomially; the kinds of tractability mentioned above have been most natural for such problems. E. Novak and H. Woźniakowski in their books [10–12] summarized a substantial amount of research work in this area. There has been recent work in the worst case setting (see, e.g., [1–7]) on problems having infinite smoothness, including problems defined over spaces of analytic functions. For such problem classes, the convergence rate of the minimal error sequence will often be faster than polynomial (e.g., exponential). Hence these papers used some concept of exponentially-convergent tractability (EC-tractability, for short). At the same time, similar concepts (polylog-tractability and $\ln^{\kappa}$-weak tractability) were introduced by Papageorgiou and Petras [13], also for the worst case setting. Motivated by [13], G. Xu [18] considered the average case setting for linear problems over Hilbert spaces, obtaining necessary and sufficient conditions for some kinds of EC-tractability.

In this paper, we will consider tractability and EC-tractability of a multivariate approximation problem defined over the space of continuous real functions in the average case setting. The space is equipped with a zero-mean Gaussian measure whose covariance kernel is given as a Korobov kernel with fast exponentially decaying Fourier coefficients. We give necessary and sufficient conditions for various kinds of tractability (including EC-tractability) in terms of the parameters of the problem. Although our results are proven for continuous linear functional information classes, we note that equation (5) in [9] allows us to check that our results also hold for standard information classes. Since the proof technique is very simple, we omit the details.

The paper is organized as follows. Section 2 contains basic terminology and known results. In Section 3, we study tractability in the traditional sense. Finally, we study EC-tractability in Section 4.

2. Basic concepts and known results

First, a bit of notation: we let $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ respectively denote the sets of all positive integers, non-negative integers, and integers. In addition, we define $\ln_+ x = 1 + \ln x$, $\ln^+ d = \max\{1, \ln d\}$ and $\lfloor x \rfloor$ for the integer part of x. Finally, we let $I = [0, 1]$ denote the closed unit interval.

A *continuous multivariate problem* is a sequence of solution operators $S = \{S_d : F_d \to G_d\}_{d\in\mathbb{N}}$, where F_d is a class of d-variate functions and G_d is another space for each $d \in \mathbb{N}$. To approximate such operators, we often use information-based algorithms that use finitely-many information operations. In this paper, we allow any continuous linear functional to be an information operation. The minimal number of linear functionals needed to give the solution of S_d to within ε is called the *information ε-complexity* and is denoted by $n(\varepsilon, d)$.

In this paper, we will assume that each operator $S_d : F_d \to H_d$ is a continuous linear transformation, with F_d a Banach space of d-variate real functions defined on a Lebesgue measurable set $D_d \subset \mathbb{R}^d$, and H_d is a Hilbert space with an inner product $\langle \cdot, \cdot \rangle_{H_d}$ and the induced norm $\| \cdot \|_{H_d}$ by the inner product.

For each $d \in \mathbb{N}$, we consider the approximation of $S_d(f)$ for $f \in F_d$ by using *information-based algorithms* of the form

$$A_{n,d}(f) = \phi_{n,d}(L_1(f), \dots, L_n(f)), \tag{2.1}$$

where $L_1, L_2, \dots, L_n$ are continuous linear functionals on F_d and $\phi_{n,d} : \mathbb{R}^n \to H_d$ is an arbitrary mapping. As a special case, we define $A_{0,d} = 0$.

To characterize the approximation error of $A_{n,d}$ in the average case setting, we assume that F_d is equipped with a zero-mean Gaussian measure μ_d defined on the Borel sets of F_d. Then the *average case error* of $A_{n,d}$ is defined to be

$$e(A_{n,d}) := \left(\int_{F_d} \|S_d(f) - A_{n,d}(f)\|_{H_d}^2 \mu_d(\mathrm{d}f) \right)^{1/2}.$$

Then for any $n \in \mathbb{N}_0$, the nth *minimal average case error* is defined as

$$e(n, d) := \inf_{A_{n,d}} e(A_{n,d}),$$