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On non-polynomial lower error bounds for adaptive strong approximation of SDEs☆

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ABSTRACT

Recently, it has been shown in Hairer et al. (2015) that there exists a system of stochastic differential equations (SDE) on the time interval $[0, T]$ with infinitely often differentiable and bounded coefficients such that the Euler scheme with equidistant time steps converges to the solution of this SDE at the final time in the strong sense but with no polynomial rate. Even worse, in Jentzen (2016) it has been shown that for any sequence $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$, which may converge to zero arbitrarily slowly, there exists an SDE on $[0, T]$ with infinitely often differentiable and bounded coefficients such that no approximation of the solution of this SDE at the final time based on n evaluations of the driving Brownian motion at fixed time points can achieve a smaller absolute mean error than the given number a_n. In the present article we generalize the latter result to the case when the approximations may choose the location as well as the number of the evaluation sites of the driving Brownian motion in an adaptive way dependent on the values of the Brownian motion observed so far.

1. Introduction

Let $d, m \in \mathbb{N}$, $T \in (0, \infty)$, consider a d-dimensional system of autonomous stochastic differential equations (SDE)

$$\begin{aligned} dX(t) &= \mu(X(t))\,dt + \sigma(X(t))\,dW(t), \quad t \in [0, T], \\ X(0) &= x_0 \end{aligned} \tag{1}$$

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with a deterministic initial value $x_0 \in \mathbb{R}^d$, a drift coefficient $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, a diffusion coefficient $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and an m-dimensional driving Brownian motion W, and assume that (1) has a unique strong solution $(X(t))_{t\in[0,T]}$. Our computational task is to approximate $X(T)$ by means of methods that use finitely many evaluations of the driving Brownian motion W. In particular we are interested in the following question: under which assumptions on the coefficients μ and σ exists a method of the latter type, which converges to $X(T)$ in absolute mean with a polynomial rate?

It is well-known that if the coefficients μ and σ are globally Lipschitz continuous then the classical Euler scheme achieves the rate of convergence 1/2, see [26]. Moreover, the recent literature on numerical approximation of SDEs contains a number of results on approximation schemes that are specifically designed for non-Lipschitz coefficients and achieve polynomial convergence rates for suitable classes of such SDEs, see e.g. [16,12,18,25,38,35,37,3,21,4] for SDEs with globally monotone coefficients and see e.g. [2,8,5,1,32,17,19,23,24,33,11] for SDEs with possibly non-monotone coefficients.

On the other hand, it has recently been shown in [20] that for any sequence $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$, which may converge to zero arbitrarily slowly, there exists an SDE (1) with $d = 4$ and $m = 1$ and with infinitely often differentiable and bounded coefficients μ and σ such that no approximation of $X(T)$ based on finitely many evaluations of the driving Brownian motion W converges in absolute mean faster than the given sequence $(a_n)_{n\in\mathbb{N}}$. More formally,

$$\inf_{s_1,\ldots,s_n\in[0,T]} \inf_{\substack{u\colon \mathbb{R}^n\to\mathbb{R}^4\\ \text{measurable}}} \mathbb{E}\big\|X(T) - u\big(W(s_1),\ldots,W(s_n)\big)\big\| \geq a_n. \tag{2}$$

In particular, there exists an SDE (1) with infinitely often differentiable and bounded coefficients μ and σ such that its solution at the final time cannot be approximated with a polynomial rate of convergence based on finitely many evaluations of the driving Brownian motion W. We add that the latter statement in the special case when the approximation is given by the Euler scheme with equidistant time steps has first been shown in [9].

The proof of the negative result (2) in [20] is constructive. Each of the respective SDEs is given by $X(0) = 0$ and

$$\begin{aligned} dX_1(t) &= dt,\\ dX_2(t) &= f(X_1(t))\,dW(t),\\ dX_3(t) &= g(X_1(t))\,dW(t),\\ dX_4(t) &= h(X_1(t))\cdot\cos\big(X_2(t)\cdot\psi(X_3(t))\big)\,dt \end{aligned} \tag{3}$$

for $t \in [0,T]$, where $f, g, h\colon \mathbb{R}\to\mathbb{R}$ are infinitely often differentiable, bounded and satisfy $\text{supp}(f) \subset (-\infty,\tau_1]$, $\text{supp}(g) \subset [\tau_1,\tau_2]$ and $\text{supp}(h) \subset [\tau_2, T]$ for some $0 < \tau_1 < \tau_2 < T$ and $\psi\colon \mathbb{R}\to\mathbb{R}$ is infinitely often differentiable and rapidly increasing to infinity. In particular, the difficult component X_4 of the solution of (3) at the final time is given by

$$X_4(T) = \cos\Big(\int_0^{\tau_1} f(t)\,dW(t)\cdot\psi\Big(\int_{\tau_1}^{\tau_2} g(t)\,dW(t)\Big)\Big)\cdot\int_{\tau_2}^{T} h(t)dt. \tag{4}$$

Note that the functions $f_{|[0,\tau_1]}$ and g are globally Lipschitz continuous. Hence, one can approximate both stochastic integrals appearing on the right hand side of (4) with error at most c/n based on n evaluations of the driving Brownian motion W. However, since the function ψ rapidly increases to infinity, it turns out that even when the stochastic integral of g is known explicitly, a small error in approximating the stochastic integral of f leads to a large error in approximating $X_4(T)$. In particular, the faster ψ goes to infinity the more difficult the approximation of $X_4(T)$ is.

The time points $s_1,\ldots,s_n \in [0,T]$ that are used by an approximation $u\big(W(s_1),\ldots,W(s_n)\big)$ in (2) are fixed, and therefore this negative result does not cover approximations that may choose the number as well as the location of the evaluation sites of the driving Brownian motion W in an adaptive way, e.g. numerical schemes that adjust the actual step size according to a criterion that is based on the values of the driving Brownian motion W observed so far, see e.g. [6,29,30,27,34,22,13,14] and the references therein. It is well-known that for a huge class of SDEs (1) with globally Lipschitz continuous coefficients μ and σ adaptive approximations cannot achieve a better rate of convergence compared to what is best possible for non-adaptive ones, which at the same time coincides with the best possible

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