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Orthogonal projectors onto spaces of periodic splines ☆

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ABSTRACT

The main result of this paper is a proof that for any integrable function f on the torus, any sequence of its orthogonal projections $(\widetilde{P}_n f)$ onto periodic spline spaces with arbitrary knots $\widetilde{\Delta}_n$ and arbitrary polynomial degree converges to f almost everywhere with respect to the Lebesgue measure, provided the mesh diameter $|\widetilde{\Delta}_n|$ tends to zero. We also give a new and simpler proof of the fact that the operators $\widetilde{P}_n$ are bounded on L^∞ independently of the knots $\widetilde{\Delta}_n$.

1. Introduction

1.1. Splines on an interval

In this article we prove some results about the periodic spline orthoprojector. In order to achieve this, we rely on existing results for the non-periodic spline orthoprojector on a compact interval, so we first describe some of those results for the latter operator. Let $k \in \mathbb{N}$ and $\Delta = (t_i)_{i=\ell}^{r+k}$ a knot sequence satisfying

$$\begin{aligned} &t_i \le t_{i+1}, \qquad t_i < t_{i+k}, \\ &t_\ell = \cdots = t_{\ell+k-1}, \qquad t_{r+1} = \cdots = t_{r+k}. \end{aligned}$$

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Associated to this knot sequence, we define $(N_i)_{i=\ell}^r$ as the sequence of L^∞-normalized B-spline functions of order k on Δ that have the properties

$$\operatorname{supp} N_i = [t_i, t_{i+k}],\ N_i \geq 0, \quad \sum_{i=\ell}^r N_i \equiv 1.$$

We write $|\Delta| = \max_{\ell \le j \le r}(t_{j+1} - t_j)$ for the maximal mesh width of the partition Δ. Then, define the space $\mathcal{S}_k(\Delta)$ as the set of polynomial splines of order k (or at most degree $k-1$) with knots Δ, which is the linear span of the B-spline functions $(N_i)_{i=\ell}^r$. Moreover, let P_Δ be the orthogonal projection operator onto the space $\mathcal{S}_k(\Delta)$ with respect to the ordinary (real) inner product $\langle f, g\rangle = \int_{t_\ell}^{t_{r+1}} f(x)g(x)\,\mathrm{d}x$, i.e.,

$$\langle P_\Delta f, s\rangle = \langle f, s\rangle \quad \text{for all } s \in \mathcal{S}_k(\Delta).$$

The operator P_Δ is also given by the formula

$$P_\Delta f = \sum_{i=\ell}^r \langle f, N_i\rangle N_i^*, \tag{1.1}$$

where $(N_i^*)_{i=\ell}^r$ denotes the dual basis to (N_i) defined by the relations $\langle N_i^*, N_j\rangle = 0$ when $j \neq i$ and $\langle N_i^*, N_i\rangle = 1$ for all $i = \ell, \dots, r$. A famous theorem by A. Shadrin states that the L^∞-norm of this projection operator is bounded independently of the knot sequence Δ:

Theorem 1.1 (*[8]*)**.** *There exists a constant c_k depending only on the spline order k such that for all knot sequences $\Delta = (t_i)_{i=\ell}^{r+k}$ as above,*

$$\|P_\Delta : L^\infty[t_\ell, t_{r+1}] \to L^\infty[t_\ell, t_{r+1}]\| \le c_k.$$

We are also interested in the following equivalent formulation of this theorem, which is proved in [1]: for a knot sequence Δ, let (a_{ij}) be the matrix $(\langle N_i^*, N_j^*\rangle)$, which is the inverse of the banded matrix $(\langle N_i, N_j\rangle)$. Then, the assertion of Theorem 1.1 is equivalent to the existence of two constants $K_0 > 0$ and $\gamma_0 \in (0,1)$ depending only on the spline order k such that

$$|a_{ij}| \le \frac{K_0 \gamma_0^{|i-j|}}{\max\{\kappa_i, \kappa_j\}}, \quad \ell \le i, j \le r, \tag{1.2}$$

where κ_i denotes the length of $\operatorname{supp} N_i$. The proof of this equivalence uses Demko's theorem [4] on the geometric decay of inverses of band matrices and de Boor's stability (see [2] or [5, Chapter 5, Theorem 4.2]) which states that for $0 < p \le \infty$, the L^p-norm of a B-spline series is equivalent to a weighted ℓ^p-norm of its coefficients, i.e. there exists a constant D_k depending only on the spline order k such that:

$$D_k k^{-1/p}\Big(\sum_j |c_j|^p \kappa_j\Big)^{1/p} \le \Big\|\sum_j c_j N_j\Big\|_{L^p} \le \Big(\sum_j |c_j|^p \kappa_j\Big)^{1/p}.$$

In fact, for a_{ij}, we actually have the following improvement of (1.2) (see [6]): There exist two constants $K > 0$ and $\gamma \in (0,1)$ that depend only on the spline order k such that

$$|a_{ij}| \le \frac{K\gamma^{|i-j|}}{h_{ij}}, \quad \ell \le i, j \le r, \tag{1.3}$$

where h_{ij} denotes the length of the convex hull of $\operatorname{supp} N_i \cup \operatorname{supp} N_j$. This inequality can be used to obtain almost everywhere convergence for spline projections of L^1-functions:

Theorem 1.2 (*[6]*)**.** *For all $f \in L^1[t_\ell, t_{r+1}]$ there exists a subset $A \subset [t_\ell, t_{r+1}]$ of full Lebesgue measure such that for all sequences (Δ_n) of partitions of $[t_\ell, t_{r+1}]$ such that $|\Delta_n| \to 0$, we have*

$$\lim_{n\to\infty} P_{\Delta_n} f(x) = f(x), \quad x \in A.$$

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