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Complexity of Banach space valued and parametric stochastic Itô integration[☆]

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Dedicated to the memory of Joseph F. Traub

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ABSTRACT

We present a complexity analysis for strong approximation of Banach space valued and parameter dependent scalar stochastic Itô integration, driven by a Wiener process. Both definite and indefinite integration are considered. We analyze the Banach space valued version of the Euler–Maruyama scheme. Based on these results, we define a multilevel algorithm for the parameter dependent stochastic integration problem and show its order optimality for various input classes.

1. Introduction

The complexity of stochastic integration was first investigated in [22]. The authors consider the problem of approximating stochastic Itô integrals of the form $\int_0^1 f(t, W(t))dW(t)$, where $(W(t))_{t\in[0,1]}$, $W(t) = W(t, \omega)$, denotes a standard Wiener process on a probability space $(\Omega, \Sigma, \mathbb{P})$. They studied the complexity of the problem, which depends on the smoothness of the integration function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$. For this purpose, they analyzed the Milstein scheme and provided a matching lower bound for certain problem classes. Moreover, they analyzed the Euler–Maruyama scheme and conjectured its order optimality (for certain problem classes). This conjecture was later proved to be true in [10]. The results are based on the assumption that standard information is available, i.e., evaluations of f and $W(t)$. The case of linear information was investigated in [17].

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Extending the analysis of [22], we study the complexity of definite and indefinite stochastic Itô integration of random functions $f : [0, 1] \times \Omega \to X$, with X a Banach space, thus, with $f(t) = f(t, \omega)$ we are interested in the approximation of

$$\int_0^t f(\tau)dW(\tau)$$

for all $t \in [0, 1]$ simultaneously in the indefinite case, and respectively for $t = 1$ in the definite case. Stochastic integration in X is closely connected to a geometric property of X, namely, the martingale type 2. We go beyond this class by considering functions Tf which are images of functions with values in some Banach space Y under an operator $T : Y \to X$ of martingale type 2. This is needed for our second goal in this paper: the investigation of stochastic Itô integration of parameter dependent scalar valued functions $f : Q \times [0, 1] \times \Omega \to \mathbb{R}$,

$$\int_0^t f(s, \tau)dW(\tau) \quad (s \in Q, t \in [0, 1]),$$

where $f(s, \tau) = f(s, \tau, \omega)$ and $Q = [0, 1]^d$ is the parameter domain.

The complexity of Banach space valued (non-stochastic) integration was first considered in [2], the complexity of (non-stochastic) parametric integration has been treated in [9,7,8], and also in [2]. It turned out that the consideration of Banach space valued algorithms can be crucial for the analysis of parametric problems. Here we follow the same line to derive algorithms for parametric stochastic integration and state complexity results. We define and analyze the Banach space valued versions of the Euler–Maruyama scheme. We obtain the same order of convergence as for the scalar valued case.

A similar situation occurs in the case of parameter dependent stochastic integration, where two cases have to be distinguished. In the case of higher parameter smoothness we obtain the same rate (up to logarithmic factors) as for non-parametric scalar stochastic integration. In the case of lower parameter smoothness we obtain the rate (again up to logarithmic factors) of approximation of functions depending only on the parameter—in other words, a rate as if we had full knowledge on the integrals. These improvements are achieved due to the multilevel structure of the algorithms.

The multilevel technique, used here, was first introduced for the complexity analysis in the randomized setting of problems such as global solution of integral equations in [6] and parametric integration in [9], see also [7,8]. Later such multilevel schemes were used for the approximation of quadrature problems of stochastic differential equations, see [5]. Our general multilevel algorithm is a combination of a Banach space valued algorithm and common interpolation operators connected via multilevel techniques.

We also prove lower bounds which are matching with the upper bounds resulting from the error estimates, thus showing the optimality of the algorithms and establishing the complexity of the considered problems (in some cases up to logarithmic factors).

The structure of the paper is as follows: In Section 2, we briefly introduce the needed results from probability theory and stochastic integration in Banach spaces. In Section 3, we analyze the Euler–Maruyama scheme for Banach space valued stochastic integrals, while in Section 4, we develop a general multilevel scheme in Banach spaces which is similar to the one introduced in [2,3]. We apply this algorithm to the scalar parametric case in Section 5 and finally, in Section 6, we present the complexity results for the previously considered problems.

2. Preliminaries

2.1. Notation

Let $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Let X, Y be Banach spaces. The closed unit ball of X is denoted by B_X, the dual space by X^*, the identity operator on X by I_X, and the space of bounded linear operators from Y to X by $\mathscr{L}(Y, X)$. Let $d \in \mathbb{N}$. The space of continuous functions on a compact set $Q \subset \mathbb{R}^d$ with values in X is denoted by $C(Q, X)$ and is equipped with the supremum norm. Furthermore, if Q is the closure of an open bounded set, then for $r \in \mathbb{N}$, $C^r(Q, X)$ stands for the