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Adaptive approximation of the minimum of Brownian motion[☆]

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ABSTRACT

We study the error in approximating the minimum of a Brownian motion on the unit interval based on finitely many point evaluations. We construct an algorithm that adaptively chooses the points at which to evaluate the Brownian path. In contrast to the 1/2 convergence rate of optimal nonadaptive algorithms, the proposed adaptive algorithm converges at an arbitrarily high polynomial rate.

1. Introduction

We study the pathwise approximation of the minimum

$$M = \inf_{0 \le t \le 1} W(t)$$

of a Brownian motion W on the unit interval $[0, 1]$ based on adaptively chosen function values of W. In contrast to nonadaptive algorithms, which evaluate a function always at the same points, adaptive algorithms may sequentially choose points at which to evaluate the function. For the present problem, this means that the nth evaluation site may depend on the first $n - 1$ observed values of the Brownian path W. Given a number of evaluation sites, we are interested in algorithms that have a small error in the residual sense with respect to the L_p-norm.

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A key motivation for studying this approximation problem stems from numerics for the reflected Brownian motion given by

$$\hat{W}(t) = W(t) - \inf_{0 \le s \le t} W(s).$$

Apart from its use in queueing theory [11], the reflected Brownian motion also appears in the context of nonlinear stochastic differential equations. More precisely, the solution process of a particular instance of a Cox–Ingersoll–Ross process is given by the square of $\hat{W}$. Hence numerical methods for the approximation of M can be used for the approximation of $\hat{W}$ and thus for the corresponding Cox–Ingersoll–Ross process. We refer to [7] for such an application of the algorithm proposed in this paper.

The complexity analysis of pathwise approximation of the Brownian minimum M based on finitely many function evaluations was initiated in [18], where it was shown that for any nonadaptive algorithm using n function evaluations the average error is at least of order $n^{-1/2}$. Moreover, a simple equidistant discretization already has an error of order $n^{-1/2}$, and thus achieves the lower bound for nonadaptive algorithms. A detailed analysis of the asymptotics of the pathwise error in case of an equidistant discretization was undertaken in [1].

The situation regarding adaptive algorithms for the pathwise approximation of M is rather different. In [5], it was shown that for any (adaptive) algorithm using n function evaluations the average error is at least of order $\exp(-c\, n/\log(n))$ for some positive constant c. In contrast to the nonadaptive case, we are unaware of algorithms with error bounds matching the lower bound for adaptive algorithms. In this paper we analyze an adaptive algorithm that has an average error at most of order n^{-r}, for any positive number r. Hence this algorithm converges at an arbitrarily high polynomial rate. In [6], the same algorithm was shown to converge in a probabilistic sense. We are unaware of previous results showing the increased power of adaptive methods relative to nonadaptive methods with respect to the L_p-error.

Several optimization algorithms have been proposed that use the Brownian motion as a model for an unknown function to be minimized, including [9,12,24,3]. One of the ideas proposed in [9] is to evaluate the function next at the point where the function has the maximum conditional probability of having a value less than the minimum of the conditional mean, minus some positive amount (tending to zero). This is the same idea behind our algorithm, described in Section 2. The question of convergence of such (Bayesian) methods in general is addressed in [13]. Several algorithms, with an emphasis on the question of convergence, are described in [21].

In global optimization, the function to be optimized is typically assumed to be a member of some class of functions. Often, the worst-case error of algorithms on such a class of functions is studied. However, if the function class is convex and symmetric, then the worst-case error for any method using n function evaluations is at least as large as the error of a suitable nonadaptive method using $n + 1$ evaluations, see, e.g., [14, Chap. 1.3]. In this case, a worst-case analysis cannot justify the use of adaptive algorithms for global optimization. An average-case analysis, where it is assumed that the function to be optimized is drawn from a probability distribution, is an alternative to justify adaptive algorithms for general function classes. For a comprehensive introduction to average-case analysis, including the problem of global optimization, we refer to [14,22,19]. Brownian motion is suitable for such an average-case study since its analysis is tractable, yet the answers to the complexity questions are far from obvious. As already explained, adaptive methods are much more powerful than nonadaptive methods for optimization of Brownian motion.

This paper is organized as follows. In Section 2 we present our algorithm with corresponding error bound, see Theorem 1. In Section 3 we illustrate our results by numerical experiments. The rest of the paper is devoted to proving Theorem 1.

2. Algorithm and main result

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with $f(0) = 0$. We will recursively define a sequence

$$t_0, t_1, \ldots \in [0, 1] \tag{1}$$