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Complexity of certain nonlinear two-point BVPs with Neumann boundary conditions☆

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ABSTRACT

We study the solution of two-point boundary-value problems for second order ODEs with boundary conditions imposed on the first derivative of the solution. The right-hand side function g is assumed to be r times ($r \geq 1$) continuously differentiable with the rth derivative being a Hölder function with exponent $\varrho \in (0, 1]$. The boundary conditions are defined through a continuously differentiable function f. We define an algorithm for solving the problem with error of order $m^{-(r+\varrho)}$ and cost of order $m \log m$ evaluations of g and f and arithmetic operations, where $m \in \mathbf{N}$. We prove that this algorithm is optimal up to the logarithmic factor in the cost. This yields that the worst-case ε-complexity of the problem (i.e., the minimal cost of solving the problem with the worst-case error at most $\varepsilon > 0$) is essentially $\Theta((1/\varepsilon)^{1/(r+\varrho)})$, up to a $\log 1/\varepsilon$ factor in the upper bound. The same bounds hold for $r + \varrho \geq 2$ even if we additionally assume convexity of g. For $r = 1$, $\varrho \in (0, 1]$ and convex functions g, the information ε-complexity is shown to be $\Theta((1/\varepsilon)^{1/2})$.

1. Introduction

We consider the solution of a boundary-value problem with Neumann boundary conditions

$$u''(x) = g(u(x)), \quad x \in [0, T], \qquad u'(0) = 0, \qquad u'(T) = f(u(T)), \tag{1}$$

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where $T > 0$ and $g,\ f$ are real functions of a real variable $u, u \ge 0$. We look for a nonnegative function u satisfying (1). The ODE problem (1) is related to stationary solutions of the heat PDE, which in turn models some physical, engineering processes or chemical reactions.

In general, the ε-complexity of ODEs, i.e., the minimal cost of computing an ε-approximation, is well studied in the case when initial conditions are imposed on the solution. The results are established in the deterministic, randomized and quantum settings, see e.g. [1,9,10,12] or [13]. Less is known about the solution of ODEs with boundary conditions, where most results deal with the Dirichlet case, see e.g. [11] or [7].

An efficient solution of the Neumann problem (1) under different conditions on g and f is recently a topic of a number of papers, see e.g., [5,2–4] or [6]. Some results, e.g., those in the recent paper [6], establish under certain assumptions on g and f upper bounds on the ε-complexity of (1). The authors consider in [6] function g possessing two continuous derivatives with g'' satisfying the Lipschitz condition, and a constant function f. The general idea is to formulate the problem as a nonlinear equation in a Banach space with the solution u. Then, the (abstract) Newton method is used and next discretized using a mesh independence principle. This leads to a finite dimensional problem with many variables. Taking care of the choice of an initial approximation and convergence of the Newton method, the authors arrive at an algorithm that computes an ε-approximation with cost $O((1/\varepsilon)^{1/2} \log\log 1/\varepsilon)$ evaluations of g and arithmetic operations.

In the present paper we consider functions g possessing r continuous derivatives ($r \ge 1$) with the rth derivative satisfying the Hölder condition with exponent $\varrho \in (0, 1]$. The function f is assumed to be a C^1 nonincreasing function, see the next section for precise definitions. The problem studied in [6] corresponds to the case $r = 2$, $\varrho = 1$, and a constant function f. We use a solution method based on 'shooting', i.e., on considering a basic scalar nonlinear equation (4) corresponding to (1) and a proper method for solving IVPs. This leads us to an almost optimal algorithm for solving (1) which only requires scalar computations.

The results of this paper are as follows. We define an algorithm for solving the second order IVPs (3) corresponding to (1) based on the idea of the approximate Picard iterations used e.g. in [1], and prove a convergence result in Proposition 1. Using (4), the bisection method and the algorithm for IVPs, we define the algorithm Φ_m^* for the problem (1) ($m \to \infty$). In Theorem 1 we prove upper bounds on the error and cost of Φ_m^*, both individually for given g and f and for the worst-case in the classes $G^{r,\varrho}$ and F of these functions. The error of Φ_m^* is shown to be $O(m^{-(r+\varrho)})$, and the cost $O(m \log m)$ evaluations of g, f and arithmetic operations, as $m \to \infty$.

We next study the optimality of Φ_m^*. We prove in Theorem 2 that the worst-case error in $G^{r,\varrho}$ and F of *any* algorithm using m evaluations of g and f or their derivatives is at least $\Omega(m^{-(r+\varrho)})$. For $r + \varrho \ge 2$, this holds even in a smaller class of convex functions $\hat{G}^{r,\varrho}$. For $r = 1$ and $\varrho \in (0, 1)$, in the smaller class $\hat{G}^{1,\varrho}$ we get the lower bound $\Omega(n^{-2})$. This means that Φ_m^* is optimal in the class of functions $G^{r,\varrho}$ up to the $\log m$ factor in the cost. The same holds for the class $\hat{G}^{r,\varrho}$ with $r+\varrho \ge 2$. In the case $r = 1$ and $\varrho \in (0, 1]$ in the class $\hat{G}^{1,\varrho}$ we define a modified algorithm Φ_m^{**} with error $O(m^{-2})$ and cost $O(m)$, which closes the gap between lower and upper bounds (Proposition 1a and Theorem 1a). Finally, Theorem 3 contains the resulting upper and lower ε-complexity bounds for the problem (1) in the classes $G^{r,\varrho}$ and $\hat{G}^{r,\varrho}$. They are of order $(1/\varepsilon)^{1/(r+\varrho)}$ from below, and $(1/\varepsilon)^{1/(r+\varrho)} \log 1/\varepsilon$ from above for $G^{r,\varrho}$, and for $\hat{G}^{r,\varrho}$ with $r + \varrho \ge 2$. In the class $\hat{G}^{1,\varrho}$ with $\varrho \in (0, 1]$, the information ε-complexity is $\Theta((1/\varepsilon)^{1/2})$. In Remark 2 at the end of Section 5 we give an idea how to get rid of the logarithmic factors that are present in the upper bounds on the information ε-complexity.

Let us note that the authors of [6], in the special case $r = 2$, $\varrho = 1$ and f being a constant function, obtained the complexity bound $O((1/\varepsilon)^{1/2} \log\log 1/\varepsilon)$. For the same parameters r and ϱ, under weaker assumptions on f, we get the bound $O((1/\varepsilon)^{1/3} \log 1/\varepsilon)$ (which is also shown to be the best possible up to the logarithmic factor). This is an improvement over [6], which is possible due to making use of the Lipschitz continuity of g''. See also Remark 1 at the end of Section 4.

The paper is organized as follows. Section 2 is devoted to the definitions of the classes of functions under consideration, the errors and the ε-complexity. We summarize there the main results of the paper (Theorems 1, 1a, 2 and 3). In Section 3 we define algorithms for (4) and the algorithms Φ_m^* and Φ_m^{**} for (1). Propositions 1 and 1a contain the IVPs convergence results. The proofs of Propositions 1, 1a

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