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# Product rules are optimal for numerical integration in classical smoothness spaces

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ABSTRACT

We mainly study numerical integration of real valued functions defined on the $d$-dimensional unit cube with all partial derivatives up to some finite order $r \geq 1$ bounded by one. It is well known that optimal algorithms that use $n$ function values achieve the error rate $n^{-r/d}$, where the hidden constant depends on $r$ and $d$. Here we prove explicit error bounds without hidden constants and, in particular, show that the optimal order of the error is $\min\{1, d\, n^{-r/d}\}$, where now the hidden constant only depends on $r$, not on $d$. For $n = m^d$, this optimal order can be achieved by (tensor) product rules.

We also provide lower bounds for integration defined over an arbitrary open domain of volume one. We briefly discuss how lower bounds for integration may be applied for other problems such as multivariate approximation and optimization.


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## 1. Introduction

Multivariate integration is nowadays a popular research problem especially when the number of variables $d$ is huge. In this paper we mainly study numerical integration of $r \geq 1$ times continuously differentiable periodic and nonperiodic functions defined over the $d$-dimensional unit cube whose partial derivatives up to order $r \geq 1$ are bounded by one. Already in 1959, Bakhvalov [2] proved that the minimal number $n = n(\varepsilon, d, r)$ of function values which is needed to achieve an error at most $\varepsilon > 0$ satisfies

$$c_{d,r}\varepsilon^{-d/r} \leq n(\varepsilon, d, r) \leq C_{d,r}\varepsilon^{-d/r}$$

for some positive $c_{d,r}$ and $C_{d,r}$ and the upper bounds are achieved by product rules. Note that for fixed $d$ and $r$ we have a sharp behaviour with respect to $\varepsilon^{-d/r}$ and $n(\varepsilon, d, r) = \Theta(\varepsilon^{-d/r})$.

For large $d$, we would like to know how $c_{d,r}$ and $C_{d,r}$ depend on $d$. Unfortunately up to 2014, the knowledge on the dependence on $d$ was quite limited since the known lower bound on $c_{d,r}$ was exponentially small in $d$ whereas the known upper bound on $C_{d,r}$ was exponentially large in $d$. In [4], we proved for the nonperiodic case that there exists a positive $c_r$ such that for all $d$ and $\varepsilon \in (0, 1)$ we have

$$n(\varepsilon, d, r) \geq c_r(1 - \varepsilon)\, d^{\,d/(2r+3)}. \tag{1}$$

Hence, we have a super-exponential dependence on $d$. This means that numerical integration suffers from the so-called curse of dimensionality for fixed $r$.

However, the exponent $d$ in (1) is $d/(2r + 3)$, whereas in Bakhvalov's lower bound it is larger and equals $d/r$. Furthermore, there is really no dependence on $\varepsilon^{-1}$ in (1), although we expect from Bakhvalov's bounds that it should be $\varepsilon^{-d/r}$.

This is the point of departure of the current paper. We improve the lower bound (1) and find a matching upper bound. Furthermore we will do it for the nonperiodic and periodic cases, (the periodic case was not studied in [4]). The lower bound is found in a similar way to [4] but instead of working with balls in the $\ell_2$-norm we switch to balls in the $\ell_1$-norm which yields a better result. The upper bound is achieved by product rules of $d$ copies of the rectangle (or trapezoidal) quadrature for the periodic case and of the Gaussian quadrature for the nonperiodic case.

We need a few definitions to formulate our results. We mainly study the problem of numerical integration, i.e., of approximating the integral

$$S_d(f) = \int_{[0,1]^d} f(x)\, \mathrm{d}x \tag{2}$$

for integrable functions $f\colon [0, 1]^d \to \mathbb{R}$.

The function class under consideration is the unit ball in the space of all $r$-times continuously differentiable functions on $[0, 1]^d$, i.e.,

$$\mathcal{C}_d^r = \{f \in C^r([0, 1]^d)\colon \|D^\beta f\|_\infty \leq 1 \text{ for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta|_1 \leq r\}$$

equipped with the norm

$$\|f\|_{\mathcal{C}_d^r} := \max_{\beta\colon |\beta|_1 \leq r} \|D^\beta f\|_\infty.$$

Here, $D^\beta$ denotes the usual (weak) partial derivative of order $\beta \in \mathbb{N}_0^d$. Moreover, the sup-norm of a bounded function $f$ is given by $\|f\|_\infty = \sup_{x\in[0,1]^d} |f(x)|$.

We consider algorithms for approximating $S_d(f)$ that use finitely many function values. More precisely, the general form of an algorithm that uses $n$ function values is

$$A_n(f) = \varphi_n(f(x_1), f(x_2), \ldots, f(x_n)) \quad \text{for all } f \in \mathcal{C}_d^r,$$

where $\varphi_n : \mathbb{R}^d \to \mathbb{R}$ may be a nonlinear mapping and the sample points $x_i \in [0, 1]^d$ may be chosen adaptively, that is, the choice of $x_i$ may depend on the already computed $f(x_1), f(x_2), \ldots, f(x_{i-1})$. Nonadaption means that the choice of $x_i$ is independent of $f$, i.e., it is the same for all functions $f$ from $\mathcal{C}_d^r$. Obviously even for the nonadaptive case, $x_i$ may depend on $n$ and the class $\mathcal{C}_d^r$.

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