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Complexity of linear ill-posed problems in Hilbert space ☆

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ABSTRACT

Information complexity of ill-posed problems may be seen as controversial. On the one hand side there were pessimistic results stating that the complexity is infinite, while on the other hand side the theory of ill-posed problems is well developed. In contrast to well-posed problems (continuous solution operators) the complexity analysis of ill-posed problems (discontinuous solution operators) is impossible without taking into account the impact of noise in the information. Commonly used models consider bounded deterministic noise and unbounded stochastic (Gaussian white) noise. It is common belief that white noise makes ill-posed problems more complex than problems under bounded noise. In this study we shed light on a rigorous complexity analysis of ill-posed problems providing (tight) lower and upper bounds for both noise models. It will be shown that in contrast to the deterministic case statistical ill-posed problems have finite complexity at every prescribed error level. Moreover, the ill-posedness of the problem raises the issue of adaptation to unknown solution smoothness, and we provide results in this direction.

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1. Introduction

Information-based complexity is concerned with complexity issues of numerical problems given in terms of solution operators. We highlight this as

$$S\colon Y \longrightarrow X, \tag{1}$$

and we confine here to linear solution operators acting between Hilbert spaces Y and X. In order to tackle complexity, we need to specify some input class $\mathcal{G} \subset Y$, typically a unit ball of some (compactly) embedded space $Y_1 \hookrightarrow Y$. We recommend the seminal monograph [25] for details.

The theory is well developed for continuous linear solution operators. It was thus interesting to discuss whether the theory extends to unbounded linear operators. This problem was treated in the influential paper by A. Werschulz [27] in 1987, surveyed in [26]. In that study the problem

$$S\colon \mathcal{D}om(S) \subset Y \longrightarrow X, \tag{2}$$

for some unbounded linear mapping is considered, and the class $\mathcal{G}$ of instances is

$$\mathcal{G} = \{y \in \mathcal{D}om(S),\ \ \|y\|_Y \leq 1\}.$$

Information is assumed to be any continuous linear functional. The major observation is then (Theorems 2.1, 2.2, ibid.) that the complexity is infinite, regardless whether the functionals are chosen nonadaptively or adaptively. This principal result received considerable attention.

The situation changes drastically if we impose additional knowledge on the solution element $x = S(y)$, for instance that it belongs to some compact subset $\mathcal{F} \subset X$. Actually, compactness is not necessary. This was also addressed in [26, p. 273] by arguing that "…it appears to be necessary to make additional a priori assumptions on the solution". For compact subsets $\mathcal{F}$ Tikhonov's Theorem [24] states that the ill-posedness of the original problem (2) will be *conditionally well-posed*. If this is the case then $S|_{S^{-1}(\mathcal{F})}$ is a continuous linear operator.

Remark 1. Werschulz' results were also presented in the monograph [25, Chapt. 5.7]. In a discussion paper [22] Th. Seidman tried to explain how to remedy the infinite complexity of ill-posed equations, by turning to different concepts of solutions, topologies, …In another paper [23, Section 2] the importance of *auxiliary information*, in particular in the light of Tikohonov's theorem is highlighted.

For statistical inverse problems the present authors explicitly discussed complexity issues in [16].

If we have conditional well-posedness then it is natural to consider the *modulus of continuity*, say $\tilde{\omega}$ for a moment, of the mapping $S|_{S^{-1}(\mathcal{F})}$, given as a function of $\delta > 0$ as

$$\tilde{\omega}_{S^{-1}(\mathcal{F})}(\delta) := \sup\left\{\left\|Sy - Sy'\right\|_X,\ y, y' \in S^{-1}(\mathcal{F}),\ \ \left\|y - y'\right\|_Y \leq \delta\right\}.$$

If we now introduce the operator $T = S^{-1}\colon X \to Y$, then the modulus rewrites as

$$\tilde{\omega}_{\mathcal{F}}(\delta) = \sup\left\{\left\|x - x'\right\|_X,\ x, x' \in \mathcal{F},\ \ \left\|Tx - Tx'\right\|_Y \leq \delta\right\}.$$

The continuity of the restricted mapping $S|_{S^{-1}(\mathcal{F})}$ yields that the modulus tends to zero as $\delta \to 0$. For balanced convex sets $\mathcal{F}$ this modulus reduces to

$$\omega_{\mathcal{F}}(\delta) := \sup\left\{\|x\|_X,\ x \in \mathcal{F},\ \ \|Tx\|_Y \leq \delta\right\},\quad \delta > 0. \tag{3}$$

Remark 2. Actually, for the properties of the modulus of continuity we do not need to consider *compact* operators T. Their convergence behavior extends to (bounded) non-compact operators with non-closed range. However, the subsequent results will be based on the singular value decomposition, and there compactness is crucial.

Here we shall consider sets $\mathcal{F} \subset X$ which are *ellipsoids*, i.e., the image of a unit ball under some self-adjoint positive operator $G\colon X \to X$, hence $\mathcal{F} = \{Gv,\ \ \|v\| \leq 1\}$. Furthermore, we need to specify the error criterion for any reconstruction, say $\hat{S}\colon Y \to X$.