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Very singular solution and short time asymptotic behaviors of nonnegative singular solutions for heat equation with nonlinear convection ☆

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Abstract

In this paper we study the following Cauchy problem:

$$u_t = u_{xx} + (u^n)_x, \quad (x,t) \in \mathbb{R} \times (0,\infty),$$
$$u(x,0) = 0, \qquad x \neq 0,$$

where parameter $n \geq 0$. Its nonnegative solution is called singular solution when $u(x,t)$ satisfies the equation in the sense of distribution, initial conditions in the classical sense and also $u(x,t)$ exhibits a singularity at the origin $(0,0)$. As we know, the singular solution is called source-type solution if the initial is $M\delta(x)$, where $\delta(x)$ is Dirac measure and constant $M > 0$. The solution is called very singular solution if it possesses more singularity than that of source-type solution at the origin. Here we focus on what happens in the interactive effect between the diffusion and convection in a whole physical process. We find critical values $n_2 < n_1 < n_0$ such that there exists unique source-type solution in the exponent range of $0 < n < n_0$, while there exists no nonnegative singular solution if $n \geq n_0$. Only in the case of $n_2 < n < n_1$ there exists a very singular solution, but in the case of $n \geq n_1$ or $n \leq n_2$ there is no solution that exhibits more singular than source-type solution at the origin. Furthermore we describe the short time asymptotic behavior of the singular solutions when such Cauchy problem is solvability for source-type solution or very singular solution.

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1. Introduction

In this paper we consider the singular nonnegative solutions of the heat equation with nonlinear convection

$$u_t = u_{xx} + (u^n)_x, \quad \text{in } S_T = \mathbb{R} \times (0, T), \tag{1.1}$$

where $n \geq 0$, with initial conditions

$$u(x, 0) = 0 \text{ for } x \neq 0, \tag{1.2}$$

and $u(x,t)$ may exhibit a singularity at origin $(0,0)$. This means a function $u(x,t)$ which is defined, nonnegative and continuous in $\bar{S_T} \setminus \{(0,0)\}$, satisfies (1.1) in the sense of distribution and initial condition (1.2) in the classical sense, and is uniformly bounded in x for every $t \in (0, T)$.

A typical singular solution of (1.1) is source-type solution.

Definition 1. For some constant $M > 0$, a function $u_M(x,t)$ defined in $S_T = \mathbb{R} \times (0,T)$ $(T > 0)$ is called a source-type solution of (1.1), if and only if:

(i) $u_M(x,t)$ is nonnegative, continuous in $\overline{S_T} \setminus \{(0,0)\}$ and bounded in $\bar{S_T^\tau} = \mathbb{R} \times [\tau, T]$ for any τ: $0 < \tau < T$;
(ii) $u_M(x,t)$ satisfies (1.1) in the sense of distribution and (1.2) in classical sense;
(iii) $u_M(x,0) = M\delta(x)$ in the sense of distribution, i.e.,

$$\lim_{\tau \to +0} \int_{-\infty}^{\infty} u_M(x,\tau)\eta(x)dx = M\eta(0) \tag{1.3}$$

for any $\eta(x) \in C_0^\infty(\mathbb{R})$.

Another singular solution of (1.1) is very singular solution. Due to the effect of the convection of (1.1), we introduce the following definition.

Definition 2. A function $u(x,t)$ defined in $S_T = \mathbb{R} \times (0,T)$ $(T > 0)$ is called a very singular solution of (1.1), if and only if $u(x,t)$ satisfies

(i) $u(x,t)$ is nonnegative, continuous in $\overline{S_T} \setminus \{(0,0)\}$ and bounded in $\bar{S_T^\tau} = \mathbb{R} \times [\tau, T]$ for any τ: $0 < \tau < T$;
(ii) $u(x,t)$ satisfies (1.1) for $(x,t) \in S_T^+ \cup S_T^-$ in the sense of distribution, where $S_T^+ = S_T \cap \{x > 0\}$ and $S_T^- = S_T \cap \{x < 0\}$;