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On semilinear Tricomi equations with critical exponents or in two space dimensions

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Abstract

This paper is a complement of our recent works on the semilinear Tricomi equations in [9] and [10]. For the semilinear Tricomi equation $\partial_t^2 u - t\Delta u = |u|^p$ with initial data $(u(0,\cdot), \partial_t u(0,\cdot)) = (u_0, u_1)$, where $t \geq 0$, $x \in \mathbb{R}^n$ $(n \geq 3)$, $p > 1$, and $u_i \in C_0^\infty(\mathbb{R}^n)$ $(i = 0, 1)$, we have shown in [9] and [10] that there exists a critical exponent $p_{\text{crit}}(n) > 1$ such that the solution u, in general, blows up in finite time when $1 < p < p_{\text{crit}}(n)$, and there is a global small solution for $p > p_{\text{crit}}(n)$. In the present paper, firstly, we prove that the solution of $\partial_t^2 u - t\Delta u = |u|^p$ will generally blow up for the critical exponent $p = p_{\text{crit}}(n)$ and $n \geq 2$, secondly, we establish the global existence of small data solution to $\partial_t^2 u - t\Delta u = |u|^p$ for $p > p_{crit}(n)$ and $n = 2$. Thus, we have given a systematic study on the blowup or global existence of small data solution u to the equation $\partial_t^2 u - t\Delta u = |u|^p$ for $n \geq 2$.

MSC: 35L70; 35L65; 35L67

Keywords: Tricomi equation; Critical exponent; Blowup; Global existence; Strichartz estimate

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1. Introduction

In this paper, we continue to be concerned with the global existence or blowup of solutions u to the semilinear Tricomi equation

$$\begin{cases} \partial_t^2 u - t\Delta u = |u|^p, \\ u(0,\cdot) = f(x), \quad \partial_t u(0,\cdot) = g(x), \end{cases} \tag{1.1}$$

where $t \geq 0$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ $(n \geq 2)$, $p > 1$, and $f, g \in C_0^\infty(B(0, M))$ with $B(0, M) = \{x : |x| = \sqrt{x_1^2 + \dots + x_n^2} < M\}$ and $M > 1$. For the local well-posedness and optimal regularities of solution u to problem (1.1), the readers may consult [19–22,31] and the references therein. In [9,10], we have determined a critical exponent $p_{crit}(n)$ and a conformal exponent $p_{conf}(n)$ $(> p_{crit}(n))$ for (1.1) as follows (corresponding to the case of $m = 1$ in the generalized equation Tricomi equation $\partial_t^2 u - t^m \Delta u = |u|^p$): $p_{crit}(n)$ is the positive root of the algebraic equation

$$(3n-2)p^2 - 3np - 6 = 0, \tag{1.2}$$

and $p_{conf}(n) = \frac{3n+6}{3n-2}$. It is shown in [9] that for all $n \geq 2$, the solution u of (1.1) generally blows up in finite time when $1 < p < p_{crit}(n)$, and meanwhile u exists globally when $p \geq p_{conf}(n)$ for small initial data and $n \geq 2$. In [10], we prove that the small data solution u of (1.1) exists globally when $n \geq 3$ and $p_{crit}(n) < p < p_{conf}(n)$. Therefore, collecting the results in [9,10], we have given a detailed study on the blowup or global existence of small data solution u to problem (1.1) for $n \geq 3$ except $p = p_{crit}(n)$, and for $n = 2$ with $p \geq p_{conf}(n)$ except $p_{crit}(n) < p < p_{conf}(n)$. In this paper, at first, we establish the finite time blowup result for problem (1.1) when $n \geq 2$ and $p = p_{crit}(n)$.

Theorem 1.1. *Let $n \geq 2$ and $p = p_{crit}(n)$. Suppose that the initial data $f, g \in C_0^\infty(\mathbb{R}^n)$ are non-negative and positive somewhere, then problem (1.1) admits no global solution u with*

$$u \in C^1\big([0,\infty), H^1(\mathbb{R}^n)\big) \cap C\big([0,\infty), L^2(\mathbb{R}^n)\big).$$

To state the global existence of small data solution u to problem (1.1) when $n = 2$ and $p_{crit}(n) < p < p_{conf}(n)$, it is convenient to introduce the angular mixed-norm space $L_t^q L_r^p L_\theta^2(\mathbb{R}_+^{1+2}) = \{u \in D'(\mathbb{R}_+^{1+2}) : (\int_0^\infty \big(\int_0^\infty \big(\int_0^{2\pi} |u(t, r cos\theta, r sin\theta)|^2 d\theta\big)^{\frac{p}{2}} dr\big)^{\frac{q}{p}} dt)^{\frac{1}{q}} < \infty\}$, where $x_1 = r\cos\theta$ and $x_1 = r\sin\theta$ with $r = \sqrt{x_1^2 + x_2^2} \geq 0$ and $\theta \in [0, 2\pi]$.

Theorem 1.2. *Let $n = 2$. For $p_{crit}(n) < p \leq p_{conf}(n)$, suppose that the smooth initial data (f, g) satisfy*

$$\sum_{|\alpha|\leq 1} (\|Z^\alpha f\|_{\dot{H}^s(\mathbb{R}^2)} + \|Z^\alpha g\|_{\dot{H}^{s-\frac{2}{3}}(\mathbb{R}^2)}) < \varepsilon, \tag{1.3}$$

where $\varepsilon > 0$ is a sufficiently small constant, $s = 1 - \frac{4}{3(p-1)}$, and $Z = \{\partial_1, \partial_2, x_1\partial_2 - x_2\partial_1\}$. Then problem (1.1) admits a global solution u such that for $|\alpha| \leq 1$,

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