ARTICLE IN PRESS YJDEQ:8968

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Wave breaking for the Fornberg–Whitham equation

Susanna V. Haziot

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, Wien 1090, Austria

Received 23 June 2017; revised 2 August 2017

Abstract

In this paper we consider the Fornberg–Whitham (FW) equation and we give sufficient conditions on the initial data to lead to wave-breaking.

MSC: 35Q35; 76B25

Keywords: Nonlinear waves; Wave breaking; Shallow water

1. Introduction

The Fornberg–Whitham equation

$$u_{xxt} - u_t + \frac{9}{2}u_x u_{xx} + \frac{3}{2}u u_{xxx} - \frac{3}{2}u u_x + u_x = 0 \tag{1}$$
$$u(x,0) = u_0(x)$$

was proposed as a model for shallow water waves (see [8]). For classical solutions decaying far out in space, e.g. $u(.,t) \in H^4(\mathbb{R}) \subset C^3(\mathbb{R})$ with a continuously differentiable dependence on time t, one can write (1) in the equivalent nonlocal conservation law form

$$u_t + \frac{3}{2}u u_x = (1 - \partial_x^2)^{-1}\partial_x u. \tag{2}$$

E-mail address: susanna.haziot@univie.ac.at.

http://dx.doi.org/10.1016/j.jde.2017.08.037

For $u(.,0) \in H^2(\mathbb{R})$, Kato's semi-group approach for quasi-linear hyperbolic equations [10] yields the existence of a unique solution $u \in C([0,T), H^2(\mathbb{R})) \cap C^1([0,T), L^2(\mathbb{R}))$, and if $T < \infty$, then $\limsup_{t \nearrow T} \|u(.,t)\|_{H^2(\mathbb{R})} = \infty$. Indeed, setting $X = L^2(\mathbb{R})$, $Y = H^2(\mathbb{R})$, and choosing the pseudo-differential operator to be $Q = (1-\partial_x^2)$, we can rewrite (2) as

$$u_t(x,t) = A(u)u + F(u,t)$$

with $A(u) = -\frac{3}{2}u\partial_x$, where the domain $\mathcal{D}(A(u)) = Y \subset X$, and $F(u,t) = Q^{-1}\partial_x u$. The requirements for the theorem for quasilinear equations of the general form

$$u_t + a(u)u_x = b(u)u, \quad t \geq 0,$$

are shown by Kato in [10].

Recently, Thompson and Holmes [12] proved the following blow-up criterion:

Theorem A *([12]). Let T be the maximal time of existence of the solution to (2) with initial data $u(.,0) \in H^2(\mathbb{R})$. If $T < \infty$ then*

$$\int_0^T \|u_x(x,t)\|_{L^\infty(\mathbb{R})}\,dt = +\infty.$$

The aim of this paper is to sharpen this blow-up criterion and to give sufficient conditions for the development of singularities in finite time in a solution to the (FW) equation.

2. Blow-up scenario

In this section, we will refine the blow-up criterion. Before we begin, we would like to recall that wave-breaking occurs when, if the maximal time of existence $T > 0$ is finite, the $\sup_{(t,x)\in[0,T)\times\mathbb{R}} |u(t,x)| < \infty$ whilst the $\sup_{x\in\mathbb{R}} |u_x(t,x)| \to \infty$ as $t \nearrow T$. We refer the reader to the discussions in [14] and [7].

Proposition 1. *If for some smooth initial data $u_0 \in H^\infty(\mathbb{R})$ the maximal existence time $T > 0$ is finite, then (2) blows-up in finite time if and only if*

$$\liminf_{t \nearrow T} [u_x(t,x)] = -\infty$$

Proof. For simplicity of notation, we denote the operator $(1-\partial_x^2)^{-1}$ by Θ. Note that this is an isomorphism between $H^s(\mathbb{R})$ and $H^{s+2}(\mathbb{R})$ for every $s \in \mathbb{R}$.

Kato's semigroup theory ensure that $u \in C^1([0,T), H^s(\mathbb{R})$ for every $s \geq 2$.

We will use the classical energy method in this proof. We begin by multiplying (2) by u and integrating over $\mathbb{R}$. We get:

$$\partial_t\{\frac{1}{2}\int_{\mathbb{R}} u^2\,dx\} = \int_{\mathbb{R}} u\Theta_x u\,dx \tag{3}$$

Please cite this article in press as: S.V. Haziot, Wave breaking for the Fornberg–Whitham equation, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.08.037