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Time periodic strong solutions to the incompressible Navier–Stokes equations with external forces of non-divergence form

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Abstract

We discuss the time periodic problem to the incompressible Navier–Stokes equations on the whole space $\mathbb{R}^n$, $n \geq 3$, with the external forces of non-divergence form. Firstly, we consider the existence of time periodic solutions in $BC(\mathbb{R}; L^{n,\infty}(\mathbb{R}^n))$ assuming the smallness of external forces in $BC(\mathbb{R}; L^1(\mathbb{R}^3))$ and $BC(\mathbb{R}; L^{\frac{n}{3},\infty}(\mathbb{R}^n))$ in the case $n \geq 4$. Next, we show that the *mild* solution above becomes a strong solution in the topology of $L^{n,\infty}(\mathbb{R}^n)$ with a natural condition of the external force, derived from the strong solvability of the inhomogeneous Stokes equations in $L^{n,\infty}(\mathbb{R}^n)$. For this aim, we re-construct a strong solvability of an abstract evolution equation where the associated semigroup is not strongly continuous at $t = 0$.

MSC: 35Q30; 76D05

Keywords: Time periodic solution; Strong solution; Lorentz space

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1. Introduction

This paper is devoted to the study for time periodic solutions to the incompressible Navier–Stokes equation on the whole space $\mathbb{R}^n$, $n \geq 3$,

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla\pi = f & \text{in } \mathbb{R}^n \times (-\infty, \infty), \\ \text{div } u = 0 & \text{in } \mathbb{R}^n \times (-\infty, \infty). \end{cases} \tag{N-S}$$

Here, $u = u(x,t) = \big(u_1(x,t), \dots, u_n(x,t)\big)$ and $\pi = \pi(x,t)$ are the unknown velocity and the pressure of the incompressible fluid, respectively, and $f = f(x,t) = \big(f_1(x,t), \dots, f_n(x,t)\big)$ is the given time periodic external force.

The history of the time periodic problem of (N-S) originated from Serrin [35]. In [35], he gave a time periodic solution in a bounded domain, based on the solvability and the stability of the initial value problem of (N-S). Yudovič [42] and Prouse [33] constructed a time periodic weak solution. Further results obtained by Kaniel and Shinbrot [16], Takeshita [38], and Miyakawa and Teramoto [30], for instance, with a reproductive property of the equations. For the case of unbounded domains, Maremonti [27] firstly paved the way. He considered the problem on the whole space and constructed a strong solution for small data of the form $f = \text{curl } \psi$. See also Maremonti and Padula [28], Salvi [34].

Kozono and Nakao [19] successfully reduced this problem to one of the integral equation (IE) stated below on interval $(-\infty, t)$, where Fujita–Kato's approach can be directly applicable. Here we write

$$u(t) = \int_{-\infty}^{t} e^{(t-s)\Delta}\mathbb{P}f(s)\,ds - \int_{-\infty}^{t} e^{(t-s)\Delta}\mathbb{P}[u \cdot \nabla u](s)\,ds, \qquad t \in \mathbb{R}, \tag{IE}$$

where $\mathbb{P}$ is the Leray–Hopf, the Weyl–Helmholtz or the Fujita–Kato projection. Controlling time integrability of both $s = -\infty$ and $s = t$, they [19] require that f has a derivative form like $f = (-\mathbb{P}\Delta)^{\delta} g$ in three dimensional case. For this direction, see also Kubo [22], Kyed [23], Farwig and Nakatsuka and Taniuchi [4], Lemarié-Rieusset [26].

Here, we refer to the stationary problem, as a special case of the time periodic problem. Finn [8] obtained the stationary solution in an exterior domain decaying like $1/|x|$ at spacial infinity, which is called a P.R. (physically reasonable) solution. The suitable framework containing such a function $1/|x|$ is $L^{n,\infty}(\mathbb{R}^n)$, in stead of $L^n(\mathbb{R}^n)$. Indeed, introducing *weak mild* solution of (N-S), Yamazaki [41] proved the existence and the stability of the time periodic solutions in $L^{n,\infty}(\mathbb{R}^n)$ with small $f = \nabla \cdot F$, $F \in L^{n/2,\infty}(\mathbb{R}^n)$. By a different approach, Galdi and Sohr [10] proved the existence time periodic solutions, behave like $1/|x|$ as $|x| \to \infty$, with small $f = \nabla \cdot F$. Furthermore, as a related result, Kang, Miura and Tsai [15] obtained the asymptotic stability of a time periodic solution around a Landau solution, see [24] and [40]. On the other hand, the uniqueness of time periodic solutions in the Lorentz spaces was investigated by Taniuchi [39], Farwig and Taniuchi [6], and Farwig and Nakatsuka and Taniuchi [5]. Recently, Geissert and Hieber and Nguyen [11] consider the time periodic problem of an abstract equation on the real interpolation spaces between abstract Banach spaces, as a generalization of this problem to the Navier–Stokes equations on the Lorentz spaces.

This paper is twofold. One is to give time periodic solutions, in integral forms, with small data of non-divergence form, and then we show that faster decay of data at infinity yields that

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