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Friedrichs systems in a Hilbert space framework: Solvability and multiplicity ☆

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Abstract

The Friedrichs (1958) theory of positive symmetric systems of first order partial differential equations encompasses many standard equations of mathematical physics, irrespective of their type. This theory was recast in an abstract Hilbert space setting by Ern, Guermond and Caplain (2007), and by Antonić and Burazin (2010). In this work we make a further step, presenting a purely operator-theoretic description of abstract Friedrichs systems, and proving that any pair of abstract Friedrichs operators admits bijective extensions with a signed boundary map. Moreover, we provide sufficient and necessary conditions for existence of infinitely many such pairs of spaces, and by the universal operator extension theory (Grubb, 1968) we get a complete identification of all such pairs, which we illustrate on two concrete one-dimensional examples.

MSC: 35F45; 46C05; 46C20; 47A05; 47B38

Keywords: Symmetric positive first-order system of partial differential equations; Kreĭn space; Universal parametrisation of extensions

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1. Introduction

Following his research on symmetric hyperbolic systems [19], Friedrichs [20] introduced the concept of 'positive symmetric system', today customarily referred to as the *Friedrichs system*, encompassing a wide variety of equations of mathematical physics, including classical elliptic, parabolic and hyperbolic equations, which can be adapted, or rewritten, in the required form.

More precisely, for a given open and bounded set $\Omega \subseteq \mathbb{R}^d$ with Lipschitz boundary Γ, let the matrix functions $\mathbf{A}_k \in \mathrm{W}^{1,\infty}(\Omega; \mathrm{M}_r)$ and $\mathbf{C} \in \mathrm{L}^{\infty}(\Omega; \mathrm{M}_r)$ satisfy $\mathbf{A}_k = \mathbf{A}_k^*$ and

$$(\exists\, \mu_0 > 0) \quad \mathbf{C} + \mathbf{C}^* + \sum_{k=1}^{d} \partial_k \mathbf{A}_k \geqslant 2\mu_0 \mathbf{I} \qquad \text{a.e. on } \Omega\,.$$

Then the first-order differential operator $T : \mathrm{L}^2(\Omega)^r \longrightarrow \mathcal{D}'(\Omega)^r$ defined by

$$T\mathsf{u} := \sum_{k=1}^{d} \partial_k(\mathbf{A}_k \mathsf{u}) + \mathbf{C}\mathsf{u}$$

is called *the (classical) Friedrichs operator* or *the symmetric positive operator*, while (for given $\mathsf{f} \in \mathrm{L}^2(\Omega)^r$) the first-order system of partial differential equations $T\mathsf{u} = \mathsf{f}$ is called *the (classical) Friedrichs system* or *the symmetric positive system*.

However, as pointed out explicitly by Friedrichs himself in [20], the main motivation of his approach 'was not the desire for a unified treatment of elliptic and hyperbolic equations, but the desire to handle equations which are partly elliptic, partly hyperbolic, such as the Tricomi equation'. Friedrichs devised a clever way of representing different boundary (or initial) conditions by using a matrix field on the boundary, which – unfortunately – was not intrinsic (i.e., unique) for a given set of boundary conditions. Finally, he was only able to prove the existence of weak solutions, and the uniqueness of strong ones, leaving the general question open on the joint existence and uniqueness of either a weak or a strong solution.

A number of improvements were made to the theory, and in some special examples the gap towards the joint existence and uniqueness was closed, mostly by Friedrichs' collaborators and former students in the following years. Whereas there was some progress in very specific points, the topic appeared to be less active from mid 1960s to the late 1990s.

New interest in Friedrichs systems arose from numerical analysis (see, for example, [24]), thanks to their feature of providing a convenient unified framework for numerical solutions to partial differential equations of different type (a comprehensive overview from this perspective can be found in [25]). In turn, this prompted further theoretical investigations of the properties of Friedrichs systems [15]. In [18] an abstract Hilbert space approach was introduced, with an intrinsic formulation of the boundary conditions, followed by an extensive study in the course of the last ten years. Three different, albeit equivalent, abstract representations of boundary conditions were motivated by their classical analogues (as proposed by Friedrichs [20], Friedrichs and Lax [21], or Phillips and Sarason [29]). Besides the well-posedness result (Theorem 5 below), important recent results include the equivalence of different representations of boundary conditions [2], its relationship with the classical theory [3–6], applications to various (initial-)boundary value problems of elliptic, hyperbolic, and parabolic type [7,12,14,15,28], and the development of different numerical schemes [10,11,13,15–17].

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