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# Emergent behaviors of the Schrödinger–Lohe model on cooperative-competitive networks

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## Abstract

We present several sufficient frameworks leading to the emergent behaviors of the coupled Schrödinger–Lohe (S–L) model under the same one-body external potential on cooperative-competitive networks. The S–L model was first introduced as a possible phenomenological model exhibiting quantum synchronization and its emergent dynamics on all-to-all cooperative networks has been treated via two distinct approaches, Lyapunov functional approach and the finite-dimensional reduction based on pairwise correlations. In this paper, we further generalize the finite-dimensional dynamical systems approach for pairwise correlation functions on cooperative-competitive networks and provide several sufficient frameworks leading to the collective exponential synchronization. For small systems consisting of three and four quantum subsystem, we also show that the system for pairwise correlations can be reduced to the Lotka–Volterra model with cooperative and competitive interactions, in which lots of interesting dynamical patterns appear, e.g., existence of closed orbits and limit-cycles.


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*Keywords:* Correlation function; Emergence; Lotka–Volterra equations; Kuramoto model; Schrödinger–Lohe system; Quantum synchronization

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**Contents**



## 1. Introduction

Collective synchronous behaviors are ubiquitous in biological and physical complex systems, e.g., the flashing of fireflies, clapping of hands in a concert hall, heartbeat regulation by pacemaker cells and arrays of Josephson junctions, etc. [1,4,5,25,26]. The research on the synchronization of weakly coupled phase oscillators has first initiated by Winfree and Kuramoto in [19,20,28] based on continuous dynamical systems about half century ago. After their pioneering works, many mathematical models were proposed and studied in literature. Among them, our interest lies on the S–L model introduced by M. Lohe [22] (see [14] for a brief survey). In this paper, we are mainly interested in the collective dynamics of S–L oscillators on networks undergoing cooperative and competitive interactions. We refer to [2,3,6,7,17] for the dynamic behaviors for S–L oscillators on an all-to-all network. So far, only attractive and all-to-all network topology has been employed in the study of the S–L oscillator synchronization. To fix the idea, we consider a network $(\mathcal{N}, \mathcal{A})$ consisting of $N$-nodes called $1, \cdots, N$ and communication(or interaction) matrix $\mathcal{A} = (a_{ij})$. The element $a_{ij}$ in $\mathcal{A}$ represents the amount (weight) of information(communication, interaction) from the $j$-th vertex to the $i$-th vertex. For example, for a positive value $a_{ij}$, the interaction between quantum systems located at $i$ and $j$ nodes is cooperative (excitatory, attractive), whereas for a negative value, the interaction is competitive (inhibitory, repulsive).

Suppose that quantum sub-systems are located at each node, and let $\psi_i = \psi_i(x,t)$ be the wave function of the $i$-th S–L oscillator on a periodic spatial domain $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$, and its dynamics is governed by the Cauchy problem to the S–L model: for $i = 1, \cdots, N$,

$$
\begin{aligned}
&\mathrm{i}\partial_t \psi_i = -\Delta\psi_i + V_i(x)\psi_i + \frac{\mathrm{i}\kappa}{N}\sum_{k=1}^{N} a_{ik}\left(\psi_k - \langle\psi_i, \psi_k\rangle\psi_i\right), \quad (x,t) \in \mathbb{T}^d \times \mathbb{R}_+, \\
&\psi_i(x,0) = \psi_i^0(x), \quad x \in \mathbb{T}^d, \quad ||\psi_i^0||_{L^2} = 1,
\end{aligned}
\tag{1.1}
$$

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