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Coupled Lane–Emden–Klein–Gordon–Fock system with central symmetry: Symmetries and conservation laws

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Abstract

A complete symmetry group classification of a coupled Lane–Emden–Klein–Gordon–Fock system with central symmetry is carried out. This system is a natural two-components generalization of the Klein–Gordon–Fock system with central symmetry. Moreover, conservation laws for the system are obtained. Some interesting physical conclusions related to the derived conserved vectors are discussed.

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Keywords: Symmetries; Conservation laws; Lane–Emden–Klein–Gordon–Fock system

1. Introduction

The class of equations of the form

$$u_{tt} - u_{rr} - \frac{n}{r}u_r + \frac{b}{r^n}u^p = 0, \tag{1.1}$$

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where b, n and p are real constants arise in many branches of mathematical physics. For example, when $n = 2$, $p = 1$ equation (1.1) is called the Klein–Gordon–Fock equation with central symmetry [1–3]. Actually, if $n = 2$, $p = 1$ in (1.1), the transformation

$$u(t,r) = \frac{\rho(t,r)}{r}, \tag{1.2}$$

transformations equation (1.1) into

$$\rho_{tt} - \rho_{rr} + \frac{b}{r^2}\rho = 0. \tag{1.3}$$

Equations (1.1) and (1.3) occur very often in several classical and quantum physical problems with central symmetry. Equation (1.3), appears in the study of electromagnetics to describe a time evolution electromagnetic fields in the homogeneous media and biconical transmission lines. Inspired by the recent work in [1–5], we consider a natural extension of equation (1.1), which we shall refer to as the coupled Lane–Emden–Klein–Gordon–Fock system with central symmetry, viz.,

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{n}{r}u_r + \dfrac{\gamma v^q}{r^n} = 0, \\ v_{tt} - v_{rr} - \dfrac{n}{r}v_r + \dfrac{\alpha u^p}{r^n} = 0, \end{cases} \tag{1.4}$$

where the field variables $u = u(t,r)$, $v = v(t,r)$ and γ, q, p and α are non-zero constants. If the parameters $n = 0$ and $\gamma = \alpha = 1$, then system (1.4) reduces to the Lane–Emden system [4] under the complex transformation $(x, y, u, v) \mapsto (t, ir, u, v)$ into the original variables of the mentioned reference. In the case when $n = 2$ and $\gamma = \alpha = 1$ in (1.4), this has been reported in [5]. For these reasons, we restrict $n \neq 0, 2$. In these classification, the nonlinearity powers will be restricted to $p, q \neq 0, 1$, so that all cases of the linear systems are excluded. The main purpose of this paper is to present classification of conservation laws and symmetries admitted by the coupled Lane–Emden–Klein–Gordon–Fock system.

The structure of this paper is two fold. In Section 2, we apply the classical Lie method [6] to set up the standard determining equations for finding Lie symmetries admitted by the coupled system. In Section 3, we apply the Noether algorithm [4,7–9] to set up the standard determining equations for finding Noether symmetries admitted by the system and thereafter construct the associated conservation laws for the underlying system. Finally, some concluding remarks are made in Section 4.

2. Lie point symmetries classification

The vector field of the form

$$X = \xi^1(t,r,u,v)\frac{\partial}{\partial t} + \xi^2(t,r,u,v)\frac{\partial}{\partial r} + \eta^1(t,r,u,v)\frac{\partial}{\partial u} + \eta^2(t,r,u,v)\frac{\partial}{\partial v}, \tag{2.5}$$