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A free boundary problem on three-dimensional cones

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Abstract

We consider a free boundary problem on cones depending on a parameter c and study when the free boundary is allowed to pass through the vertex of the cone. We show that when the cone is three-dimensional and c is large enough, the free boundary avoids the vertex. We also show that when c is small enough but still positive, the free boundary is allowed to pass through the vertex. This establishes 3 as the critical dimension for which the free boundary may pass through the vertex of a right circular cone. In view of the well-known connection between area-minimizing surfaces and the free boundary problem under consideration, our result is analogous to a result of Morgan that classifies when an area-minimizing surface on a cone passes through the vertex.

MSC: 35R35; 35R01; 35J20; 49N60

1. Introduction

We study solutions to the problem

$$\begin{aligned} \Delta u &= 0 \qquad \text{in } \{u > 0\} \\ |\nabla u| &= 1 \qquad \text{on } \partial\{u > 0\}, \end{aligned} \tag{1.1}$$

on right circular cones in $\mathbb{R}^n$. We are interested in determining when the free boundary $\partial\{u > 0\}$ is allowed to pass through the vertex of the cone.

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The above problem has applications to two dimensional flow problems as well as heat flow problems (see [4] where (1.1) was first studied). When considering the applications on a manifold, one studies a variable coefficient problem in divergence form:

$$\begin{aligned} \partial_j(a^{ij}(x)u_i) &= 0 \qquad \text{in } \{u>0\} \\ a^{ij}(x)u_iu_j &= 1 \qquad \text{on } \partial\{u>0\}. \end{aligned} \tag{1.2}$$

Solutions of (1.2) may be found inside a bounded domain Ω by minimizing the functional:

$$\int_\Omega a^{ij}(x)v_iv_j + Q(x)\chi_{\{v>0\}}. \tag{1.3}$$

However, since the functional is not convex, minimizers of (1.3) may not be unique and there exist solutions to (1.2) which are not minimizers of (1.3). When the coefficients $a^{ij}(x)$ are Lipschitz continuous and satisfy an ellipticity condition, regularity of the free boundary was studied in [15]. The authors in [15] adapted the sup-convolution approach of Caffarelli in [7–9] for viscosity solutions. This approach relies on a nondivergence structure and therefore requires Lipschitz continuity of the coefficients $a^{ij}(x)$ so that (1.3) can be transformed into a nondivergence operator. More recently, the regularity of the free boundary for Hölder continuous coefficients $a^{ij}(x)$ was accomplished in [11] using different techniques. For coefficients $a^{ij}(x)$ assumed merely to be bounded, measurable, and satisfying the usual ellipticity conditions, regularity of the solution and its growth away from the free boundary was studied in [14]. However, to date nothing is know regarding the regularity of the free boundary when the coefficients $a^{ij}(x)$ are allowed to be discontinuous. In this paper we are interested in how the free boundary interacts with isolated discontinuous points of the coefficients $a^{ij}(x)$. In the context of a hypersurface, these points are considered to be a topological singularity. The simplest such case is the vertex of a cone. The aim of this paper is to study when the free boundary of a solution that arises as a minimizer is allowed to pass through a topological singularity. Before stating the main results of this paper we first recall a connection between solutions to (1.1) and minimal surfaces in order to understand what results one might expect for the free boundary problem on a cone.

1.1. Connection to minimal surfaces

Results for the singular set of free boundary points are analogous to results for the singular set of minimal surfaces. In the case of area-minimizing surfaces, the study of the singular set is reduced to considering area-minimizing cones. Simons [20] showed that any area-minimizing cone in $\mathbb{R}^n$ for $n \le 7$ is necessarily planar. Simons actually proved a stronger result in [20] by showing that any minimal stable cone is planar. He also provided an example of a cone in $\mathbb{R}^8$ that is stable and therefore a possible candidate for being an area-minimizing cone. One year later, it was shown that the Simons cone is indeed area-minimizing, see [6]. As a consequence, $n = 8$ is the first dimension for which a singularity of an area-minimizing hypersurface may occur.

Regarding the singular set of the free boundary for minimizers, the authors in [4] showed there are no singular points in dimension $n = 2$. In [22] a monotonicity formula is utilized to show that blow-up solutions are homogeneous, and therefore the free boundary of blow-up solutions is a cone. As a further consequence there exists a minimal dimension k^* such that the singular set of the free boundary of minimizers is empty if the dimension $n < k^*$. The authors

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