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On the initial-boundary value problem for some quasilinear parabolic equations of divergence form

Mitsuhiro Nakao

Faculty of Mathematics, Kyushu University, Moto-oka 744, Fukuoka 819-0395, Japan

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Abstract

In this paper we give an existence theorem of global classical solution to the initial boundary value problem for the quasilinear parabolic equations of divergence form $u_t - \mathrm{div}\{\sigma(|\nabla u|^2)\nabla u\} = f(\nabla u, u, x, t)$ where $\sigma(|\nabla u|^2)$ may not be bounded as $|\nabla u| \to \infty$. As an application the logarithmic type nonlinearity $\sigma(|\nabla u|^2) = \log(1 + |\nabla u|^2)$ which is growing as $|\nabla u| \to \infty$ and degenerate at $|\nabla u| = 0$ is considered under $f \equiv 0$.

MSC: 35B35; 35B45; 35K20; 35K59; 35K65; 35K92

Keywords: Quasilinear parabolic equation; Growing nonlinearity; Moser's method

1. Introduction

In this paper we consider the initial-boundary value problem of the quasilinear parabolic equation of the form:

$$u_t - \mathrm{div}\{\sigma(|\nabla u|^2)\nabla u\} = f(\nabla u, u, x, t) \text{ in } \Omega \times (0, \infty), \tag{1.1}$$

with the initial-boundary conditions

E-mail address: mnakao@math.kyushu-u.ac.jp.

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$$u(x,0) = u_0(x) \text{ and } u(x,t)|_{\partial\Omega} = 0, \tag{1.2}$$

where Ω is a bounded domain in R^N with $C^{2,\alpha}, \alpha > 0$, class boundary $\partial\Omega$. We assume that $\sigma = \sigma(v^2)$ is a $C^{1,\alpha}$ class positive function satisfying the following conditions:

Hyp. A.

$$(1) \quad \sigma(v^2) + 2\sigma'(v^2)v^2 \geq k_0\sigma(v^2), \qquad (2) \quad \sigma(v^2) \geq \epsilon_0 > 0,$$

$$(3) \quad |\sigma'(v^2)v^2| \leq k_1\sigma(v^2) \quad \text{and} \quad (4) \quad k_0\sigma(v^2)v^2 \leq \int_0^{v^2} \sigma(\eta)d\eta$$

with some $\epsilon_0, k_0, k_1 > 0$.

We do not assume the boundedness of $\sigma(v^2)$ as v^2 to ∞ and we can give $\sigma = (v^2 + \epsilon)^{m/2}$, $m \geq 0$, and $\log(1 + \epsilon + v^2)$, $\epsilon > 0$, as typical examples.

For the force term we assume:

Hyp. B. $f(\nabla u, u, x, t)$ is a $C^\alpha(R^{n+1} \times \bar{\Omega} \times [0,\infty))$ class function with

$$\sup_{\bar{\Omega}\times[0,T]} |f(\nabla u, u, x, t)| \leq M_T(1 + |\nabla u|) < \infty$$

for any $0 < T < \infty$, where M_T is a constant possibly depending on T. Further, $f(\nabla u, u, x, t)$ is Lipshitz continuous in $(\nabla u, u)$ uniformly on $B^{n+1} \times \Omega \times [0, T])$ for any bounded set $B^{n+1} \subset R^{n+1}$.

Concerning the initial data we assume $u_0 \in C_0^{2,\alpha}(\Omega), \alpha > 0$.

In the famous and standard text book [6] by Ladyzenskaya, Solonnikov and Uraltseva it is proved that (in addition to the conditions (1)–(4)) if σ is bounded from above, that is, $\sigma(v^2) \leq k_1 < \infty$, the problem admits a unique classical solution $u(t) \in C^{1,\alpha/2}([0,\infty); C(\bar{\Omega})) \cap C([0,\infty); C^{2,\alpha}(\bar{\Omega}))$ (see Theorem 4.2 in Chap. V). In fact more general equations are considered there, but the conditions on σ in the above seem to be essential. Further, in [6] it is proved that if σ has a polynomial growth order, that is, $k_0(1 + |v|)^m \leq \sigma(v^2) \leq k_1(1 + |v|)^m, m \geq 0$, then the problem admits a unique classical solution (see Theorem 4.1 in Chap. VI). In the present note we start from Theorem 4.2, chap. V, in [6] and prove the existence of classical solution of the problem (1.1)–(1.2). The key step is the derivation of the boundedness of $\|\nabla u(t)\|_\infty, 0 \leq t \leq T$ for the approximate solutions $u(t)$. For this we employ Moser's technique as in [1] and [10]. In the proof of Theorem 4.1, Chap. VI, in [6] very skilful method based on maximum principle is used to derive the a priori estimate for $\|\nabla u(t)\|_\infty$ and our method is different from it. We do not make any specified assumption on the growth order and our result includes, for an example, $\sigma = \log(1 + \epsilon + |\nabla u|^2), \epsilon > 0$, which does not seem to be included in [6].

As an application we consider in the second part the degenerate case $\sigma(v^2) = \log(1 + v^2)$ with $f = 0$. That is,

$$u_t - \text{div}\{\log(1 + |\nabla u|^2)\nabla u\} = 0 \text{ in } \Omega \times (0,\infty), \tag{1.3}$$

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