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Existence and asymptotic behavior of nontrivial solutions to the Swift–Hohenberg equation

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Abstract

In this paper, we discuss several results regarding existence, non-existence and asymptotic properties of solutions to $u'''' + qu'' + f(u) = 0$, under various hypotheses on the parameter q and on the potential $F(t) = \int_0^t f(s)\,ds$, generally assumed to be bounded from below. We prove a non-existence result in the case $q \le 0$ and an existence result of periodic solution for: 1) almost every suitably small (depending on F), positive values of q; 2) all suitably large (depending on F) values of q. Finally, we describe some conditions on F which ensure that some (or all) solutions u_q to the equation satisfy $\|u_q\|_\infty \to 0$, as $q \downarrow 0$.

MSC: 34C25; 35A24; 35B38; 65L03

Keywords: Critical point theory; Higher-order ordinary differential equations; Swift–Hohenberg equation

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1. Introduction and main results

We are interested in solutions to the ordinary differential equation

$$u'''' + qu'' + F'(u) = 0, \tag{1.1}$$

where F is a smooth (say, C^2) potential which we can freely assume to satisfy $F(0) = 0$. For $q > 0$, (1.1) is the Swift–Hohenberg (briefly, S–H) equation while, for $q \le 0$, it is known as the Extended Fisher–Kolmogorov (briefly, EFK) equation. Both equations have a large number of applications, which differ substantially according to various ranges of the parameter q. We refer to the monograph [17] for its various physical derivations and many qualitative results. Historically, (1.1) was first considered for the double-well potential $F(t) = (1 - t^2)^2$, due to its relevance in phase transition problems. Heteroclinic or homoclinic solutions have been obtained in [3,15,17,19] for S–H and in [8,9,17] for EFK. Roughly speaking, the EFK case has proven to be much more manageable than the S–H case, with some basic questions still left open for the latter. After the seminal works of McKenna, Lazer and Walter (see [11,13,14]), the study of (1.1) for convex, coercive potentials became a major tool to understand the modeling of suspension bridges. Since then, much more refined, higher dimensional models have been developed, and a rather exhaustive exposition on the subject can be found in the monograph [5]. In this paper we will focus on quasi-convex potentials F, i.e., those satisfying

$$F'(t)t \ge 0, \qquad \forall t \in \mathbb{R}. \tag{1.2}$$

Under this assumption, our main interest will be answering to the following questions, mainly motivated by some open problems described in [12,16]:

1. Under what condition does (1.1) possess nontrivial (i.e., non-constant) global/bounded solutions?
2. What is the behavior, as $q \to 0$ (i.e. transitioning from S–H to EFK), of such solutions?

Regarding question 1, we are going to prove a nonexistence result (Theorem 1.5) for the EFK equation through energy methods and a Liouville type theorem proved in [16], while in the S–H case we construct global solutions both for small (Theorem 1.8) and large (Theorem 1.11) velocities, respectively through a Mountain Pass technique and the direct method of Calculus of Variations.