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Inhomogeneous oscillatory integrals and global smoothing effects for dispersive equations

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Abstract

We study oscillatory integrals of the type $\mathcal{F}^{-1}(e^{ita(\cdot)}\psi(\cdot))$ where a is a general function satisfying some elliptic type and non-degenerate conditions at both the origin and the infinity, and ψ belongs to some symbol class. Point-wise estimates in space–time are gained with partial sharpness. As applications, global smoothing effects of L^p–L^q as well as Strichartz type for dispersive equations are studied. An application to fractional Schrödinger equations is also given.

Keywords: Oscillatory integrals; Dispersive equations; L^p–L^q estimates

1. Introduction

This paper is a following study of Kenig, Ponce and Vega [16]. There, the local and global smoothing effects for a class of dispersive Cauchy problems

$$\begin{cases} \partial_t u - ia(D)u = 0, & x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

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were studied, where $D = -i\nabla_x$ and $a(D)$ is defined through its Fourier symbol $a(\xi)$. Focusing on the global smoothing effects, it was shown, for example in dimension one, that if $a(\xi)$ is a real polynomial of degree $m \geq 2$,

$$W_\gamma(t)u_0(x) := \int_{\mathbb{R}} e^{i(ta(\xi)+x\xi)}|a''(\xi)|^{\frac{\gamma}{2}}\hat{u}_0(\xi)d\xi, \quad (x,t) \in \mathbb{R}^2, \tag{1.2}$$

where $\gamma \geq 0$ and $\hat{u}_0$ denotes the Fourier transform of u_0, then for any $\theta \in [0, 1]$ we have

$$||W_{\theta/2}(t)u_0||_{L_t^q(\mathbb{R};\ L_x^p)} \leq C||u_0||_{L_x^2}. \tag{1.3}$$

By interpolation, the key ingredient to prove (1.3) is the following dispersive estimate:

$$||W_1(t)u_0||_{L_x^\infty} \leq C|t|^{-\frac{1}{2}}||u_0||_{L_x^1}, \tag{1.4}$$

which roughly means that the solution u to (1.1) has $\frac{m-2}{2}$ derivatives in L_x^∞ if $u_0 \in L_x^1$. The proof is to show that the "$|a''|^{1/2}$-derivative" of the convolution kernel of $W_1(t)$, appearing as an oscillatory integral, has the decay $|t|^{-1/2}$ uniformly in the space variable. This result was also generalized for a class of phase functions a in [16], but not in the higher dimensional version, which would be much harder and was also established there, while a is assumed to be a real polynomial of degree $m \geq 2$ having (non-degenerate) elliptic principle part, a'' in (1.2) is replaced by the Hessian Ha, and the integrand is cut off away from the origin. A special case that a is non-elliptic was also considered. We refer to [16] for the motivations and more thorough statements, as well as the local smoothing effect which is out of the scope of the present paper.

The main ingredient of this paper is to show, in a very general setting of $a(\xi)$ more than the type of polynomial, that even in higher dimensions, there are actually much more global smoothing effects for (1.1) than those associated with Ha, by studying the oscillatory integral

$$I(t,x) = \int_{\mathbb{R}^n} e^{i(ta(\xi)+x\cdot\xi)}\psi(\xi)d\xi, \quad t \in \mathbb{R}\setminus\{0\},\ x \in \mathbb{R}^n, \tag{1.5}$$

where ψ stands for the smoothing. One of the features in this study claims that, if the growth of ψ differs from that of Ha, then I may have decays both in $|t|$ and in $|x|$. This leads to more possible types of estimates for the solution u to (1.1), typically the L^p–L^q type estimates. The other important feature is the generality of a in our main result Theorem 3.1. To give a first sight, let's recall an interesting result due to Ben-Artzi, Koch and Saut [2]:

Theorem 1.1. *If $a(\xi) = |\xi|^4 + |\xi|^2$, $\psi(\xi) = \xi^\alpha$ for any multi-index α, then*

$$|I(t,x)| \leq \begin{cases} C|t|^{-\frac{n+|\alpha|}{4}}(1+|t|^{-\frac{1}{4}}|x|)^{-\frac{n-|\alpha|}{3}} & \text{if } 0 < |t| \leq 1 \text{ or } |x| \geq |t|, \\ C|t|^{-\frac{n+|\alpha|}{2}}(1+|t|^{-\frac{1}{2}}|x|)^{|\alpha|} & \text{if } |t| > 1 \text{ and } |x| < |t|. \end{cases}$$

We shall recover this result in Example 3.2. Since the phase function a is radial, in the polar coordinates, the proof in [2] uses estimates of the Bessel functions for the sphere part, and the