Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 263 (2017) 8630–8648

Journal of Differential Equations

www.elsevier.com/locate/jde

Nonuniqueness of nematic liquid crystal flows in dimension three

Huajun Gong [a], Tao Huang [b,*], Jinkai Li [c]

[a] College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China
[b] NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, 3663 Zhongshan Road North, Shanghai, 200062, China
[c] Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

Received 9 January 2017

Abstract

For suitable initial and boundary data, we construct infinitely many weak solutions to the nematic liquid crystal flows in dimension three. These solutions are in the axisymmetric class with bounded energy and "backward bubbling" at a large time.

1. Introduction

For any smooth domain $\Omega \subset R^3$, we consider the following simplified model of nematic liquid crystal flows

$$\begin{cases} u_t + u \cdot \nabla u - \mu \Delta u + \nabla P = -\lambda \nabla \cdot \left(\nabla d \odot \nabla d - \frac{1}{2} |\nabla d|^2 \mathbb{I}_3 \right), \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d = \gamma \left(\Delta d + |\nabla d|^2 d \right), \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: huajun84@hotmail.com (H. Gong), th79@nyu.edu (T. Huang), jklimath@gmail.com (J. Li).

http://dx.doi.org/10.1016/j.jde.2017.08.052

where $u(\mathbf{x},t) : \Omega \times (0,T) \to \mathbb{R}^3$ is the velocity field of the underlying incompressible fluid, $d(\mathbf{x},t) : \Omega \times (0,T) \to \mathbb{S}^2 := \{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1\}$ represents the (averaged) orientation field of nematic liquid crystal molecules, $P(\mathbf{x},t) : \Omega \times (0,T) \to \mathbb{R}$ is the pressure function, $\mathbf{x} = (x,y,z) \in \Omega$, $\nabla\cdot$ denotes the divergence operator on $\mathbb{R}^3$, $\nabla d \odot \nabla d = \left(\langle \frac{\partial d}{\partial \mathbf{x}_i}, \frac{\partial d}{\partial \mathbf{x}_j}\rangle\right)_{1\le i,j\le 3} \in \mathbb{R}^{3\times 3}$ represents the stress tensor induced by the orientation field d, and $\mathbb{I}_3 = \left(\delta_{ij}\right)_{1\le i,j\le 3} \in \mathbb{R}^{3\times 3}$ is the identity matrix of order 3. The parameters μ, λ and γ are positive constants representing the fluid viscosity, the competition between kinetic energy and potential energy, and the macroscopic elastic relaxation time for the molecular orientation field respectively. For simplicity, we only consider

$$\mu = \lambda = \gamma = 1.$$

The system (1.1) has first been proposed by Lin [29] as a simplified version of the general Ericksen–Leslie system modeling the hydrodynamic flow of nematic liquid crystal materials proposed by Ericksen [9] and Leslie [23] between 1958 and 1968. The system (1.1) is a macroscopic continuum description of the time evolution of the liquid crystal material under the influence of both the fluid field and the macroscopic description of the microscopic orientation configurations of rod-like liquid crystals molecules. The interested readers can refer to [9], [23], [29], and Lin–Liu [31] for more details. In this paper, we will investigate the system (1.1) with initial data

$$(u,d)\big|_{t=0} = (u_0, d_0)$$

and boundary data that will be specific later, where $(u_0(\mathbf{x}), d_0(\mathbf{x})) : \Omega \to \mathbb{R}^3 \times \mathbb{S}^2$ satisfies $\operatorname{div} u_0 = 0$ and $|d_0| = 1$.

Mathematically, the system (1.1) is a strong coupling between the incompressible Navier–Stokes equation for the flow field and the (transport) heat flow of harmonic maps for the orientation field of the liquid crystal molecules, which has attracted a lot of interests and generated many interesting results recently. In dimension two, Lin–Lin–Wang [30] have proved the existence of global Leray–Hopf type weak solutions to initial and boundary value problem of (1.1) with finitely many possible singular times (see [15] for $\Omega = \mathbb{R}^2$, [17,43,28] for more general systems, and [8,24,25,45] for some other related works). Lin–Wang [33], Wang–Wang–Zhang [44], and Li–Titi–Xin [27] have also proved the uniqueness for such weak solutions. In dimension three, Lin–Wang [32] have proved the existence of global weak solutions under the assumption $d_0(x) \in \mathbb{S}^2_+$ for a.e. $x \in \Omega$ by developing some new compactness arguments. Here $\mathbb{S}^2_+$ is the upper hemisphere. Huang [20] has shown the weak solution is regular and unique in the scaling invariant Leray spaces. Recently, in [21], two nontrivial examples of finite time singularities in dimension three have been constructed. The first example is built with the help of axisymmetric solutions to (1.1) without swirl. In the second example, the initial data of the approximate harmonic maps was constructed with small energy but large topology. With help of the energy inequality, the local smooth solutions have been proved to have finite time singularities by an ϵ-apriori estimate on approximate harmonic maps. However, it still remains a very challenging open problem to establish the existence of global Leray–Hopf type weak solutions and partial regularity of suitable weak solutions to (1.1) in dimension three. It should be mentioned that for suitably regular initial data, local existence and uniqueness of more regular solutions than the weak ones can be established for more general systems than (1.1), see, e.g., [16,13,14,11,12,26,37]. More results and references can be found in the survey paper by Lin–Wang [34].