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Well-posedness and attractors for a super-cubic weakly damped wave equation with H^{-1} source term

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Abstract

This work is devoted to Dirichlet problem for the sup-cubic semi-linear wave equation with H^{-1} forcing in a bounded domain of $\mathbb{R}^3$. The global well-posedness of the weak solution with some translational regularity is obtained. Moreover, the existence of a global attractor as well as exponential attractor in the natural phase space are also considered. The results are crucially based on the Strichartz estimates for the linear wave equation in bounded domain of $\mathbb{R}^3$.

Keywords: Wave equation; Sup-cubic; Strichartz estimate; Attractor

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Given $\gamma > 0$, we consider the following weakly damped wave equation:

$$\begin{cases} u_{tt} + \gamma u_t - \Delta u + f(u) = g, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0,\quad u_t(x,0) = u_1, & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.1}$$

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Here $g \in H^{-1}$ is independent of time, $f \in C^1(\mathbb{R})$ and satisfies the following conditions

$$|f'(s)| \le c_1(1+|s|^{p-1}), \tag{1.2}$$

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{1.3}$$

where $1 \le p \le \frac{25}{7}$, λ_1 is the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$.

Equation (1.1) arises as an evolutionary mathematical model in various systems for the relevant physical application including electrodynamics, quantum mechanics, nonlinear elasticity etc., and are of a big permanent interest (see e.g. [36]).

The asymptotic behavior of (1.1) has been investigated extensively by many authors (see e.g. [1–3,6,9,11,21,29,30]), which depends strongly on the growth rate of the nonlinearity f. For a long time, in a bounded smooth domain of $\mathbb{R}^3$, the global well-posedness for problem (1.1) holds only in the case of sub-cubic or cubic growth rates of nonlinearity f (that is, the exponent p in (1.2) less or equals to 3), in which the uniqueness is verified by the technology of Sobolev embedding in general. Therefore, the cubic growth rate of nonlinearity had been considered as a critical one for the case of 3-D bounded domain. Consequently, the existence of global attractors for the weakly damped wave equations in the natural energy space $H_0^1(\Omega) \times L^2(\Omega)$ as well as its regularities only had been known for the case $p \le 3$, that is, the nonlinear term f at most can be cubic growth, in 3-D bounded domain, see [1,2,4,11,16,29,36] and the references therein. Recently, with the help of the suitable versions of Strichartz estimates and generalization of Morawetz–Pohozhaev identity to the case of bounded domain, the global well-posedness of quintic wave equation in 3-D smooth bounded domains has been obtained in [5,7,8]. Based on the fact that the Strichartz estimates can be transferred from semilinear wave equation to weakly damped wave equation, Kalantarov, Savostianov and Zelik [19] obtained the well-posedness of weakly damped wave equation in the case of quintic or sub-quintic growth rates of nonlinearity f in bounded domain of $\mathbb{R}^3$, where the solution is the weak solution with extra regularity named Shatah–Struwe solution, moreover, they also considered the global attractor for the case of quintic or sub-quintic growth rates of nonlinearity f for the situation that the forcing term $g \in L^2(\Omega)$. Furthermore, in [32,33], A. Savostianov and S. Zelik analyze a more general situation, with a term of the form $(-\Delta)^\theta u_t$, for $\theta \in [0,1]$, in place of u_t.

This paper is devoted to the case $g \in H^{-1}$ and with sup-cubic nonlinearity. From the fact that there is no regularization for the wave-type equation as which holds for parabolic-type, we know that the solution u will stay in $H_0^1(\Omega)$ always, consequently for the sup-cubic case $f(u)$ can not belong to $L^2(\Omega)$, and $-\Delta u$ also only in H^{-1}, hence it is natural to consider the case $g \in H^{-1}$. In [28,35,38], the authors discussed the existence and asymptotic regularity of global attractor for the strong damping wave equations in the case of the forcing term belongs to H^{-1}, as mentioned in [38], the strong damping term $-\Delta u_t$ brings many advantages which can counterbalance the unfavorable effect of $g \in H^{-1}$. For the weakly damped wave equation (1.1), it seems difficult to apply the corresponding method to verify the well-posedness of problem (1.1) and asymptotic compactness of the corresponding solution semigroup. Just recently, in [24,26], we considered the weakly damped wave equation with H^{-1} forcing and cubic nonlinearity, by rewriting the equation as $u_{tt} + \gamma u_t - \Delta u - g(x) = -f(u)$ and observing that $f(u) \in L^2(\Omega)$, we can still multiply (1.1) by u_t directly, which is curial for uniqueness.

The case of $g \in H^{-1}$ with sup-cubic nonlinearity is indeed much more delicate. In the case of $g \in H^{-1}$, the solution of corresponding stationary equation $-\Delta h + f(h) = g$ only belongs

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