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On the existence of connecting orbits for critical values of the energy

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Abstract

We consider an open connected set Ω and a smooth potential U which is positive in Ω and vanishes on $\partial\Omega$. We study the existence of orbits of the mechanical system $\ddot{u} = U_x(u)$, that connect different components of $\partial\Omega$ and lie on the zero level of the energy. We allow that $\partial\Omega$ contains a finite number of critical points of U. The case of symmetric potential is also considered.

MSC: 37J50; 37J45

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1. Introduction

Let $U : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^2. We assume that $\Omega \subset \mathbb{R}^n$ is a connected component of the set $\{x \in \mathbb{R}^n : U(x) > 0\}$ and that $\partial\Omega$ is compact and is the union of $N \geq 1$ distinct nonempty connected components $\Gamma_1, \dots, \Gamma_N$. We consider the following situations:

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H $N \geq 2$ and, if Ω is unbounded, there is $r_0 > 0$ and a non-negative function $\sigma : [r_0, +\infty) \to \mathbb{R}$ such that $\int_{r_0}^{+\infty} \sigma(r)dr = +\infty$ and

$$\sqrt{U(x)} \geq \sigma(|x|), \quad x \in \Omega, \quad |x| \geq r_0. \tag{1}$$

H$_s$ Ω is bounded, the origin $0 \in \mathbb{R}^n$ belongs to Ω and U is invariant under the antipodal map

$$U(-x) = U(x), \quad x \in \Omega.$$

Condition (1) was first introduced in [1]. A sufficient condition for (1) is that $\liminf_{|x|\to\infty} U(x) > 0$.

We study non constant solutions $u : (T_-, T_+) \to \Omega$, of the equation

$$\ddot{u} = U_x(u), \quad U_x = \left(\frac{\partial U}{\partial x}\right)^T, \tag{2}$$

that satisfy

$$\lim_{t \to T_\pm} d(u(t), \partial\Omega) = 0, \tag{3}$$

with d the Euclidean distance, and lie on the energy surface

$$\frac{1}{2}|\dot{u}|^2 - U(u) = 0. \tag{4}$$

We allow that the boundary $\partial\Omega$ of Ω contains a finite set P of critical points of U and assume

H$_1$ If $\Gamma \in \{\Gamma_1, \dots, \Gamma_N\}$ has positive diameter and $p \in P \cap \Gamma$, then p is a hyperbolic critical point of U.

If Γ has positive diameter, then hyperbolic critical points $p \in \Gamma$ correspond to saddle-center equilibrium points in the zero energy level of the Hamiltonian system associated to (2). These points are organizing centers of complex dynamics, see [2].

Note that **H**$_1$ does not exclude that some of the Γ_j reduce to a singleton, say $\{p\}$, for some $p \in P$. In this case nothing is required on the behavior of U in a neighborhood of p aside from being of class C^2.

A comment on **H** and **H**$_s$ is in order. If P is nonempty $u \equiv p$ for $p \in P$ is a constant solution of (2) that satisfies (3) and (4). To avoid trivial solutions of this kind we require $N \geq 2$ in **H**, and look for solutions that connect different components of $\partial\Omega$. In **H**$_s$ we do not exclude that $\partial\Omega$ is connected ($N = 1$) and we avoid trivial solutions by restricting to a symmetric context and to solutions that pass through 0.

We prove the following results.

Theorem 1.1. *Assume that* **H** *and* **H**$_1$ *hold. Then for each* $\Gamma_- \in \{\Gamma_1, \dots, \Gamma_N\}$ *there exist* $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\} \setminus \{\Gamma_-\}$ *and a map* $u^* : (T_-, T_+) \to \Omega$, *with* $-\infty \leq T_- < T_+ \leq +\infty$, *that satisfies* (2), (4) *and*

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