ARTICLE IN PRESS YJDEQ:8878

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Global solutions to random 3D vorticity equations for small initial data

Viorel Barbu [a,*], Michael Röckner [b]

[a] *Octav Mayer Institute of the Romanian Academy and Al.I. Cuza University, Iaşi, Romania*
[b] *Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany*

Received 1 October 2016; revised 14 June 2017

Abstract

One proves the existence and uniqueness in $(L^p(\mathbb{R}^3))^3$, $\frac{3}{2} < p < 2$, of a global mild solution to random vorticity equations associated to stochastic $3D$ Navier–Stokes equations with linear multiplicative Gaussian noise of convolution type, for sufficiently small initial vorticity. This resembles some earlier deterministic results of T. Kato [16] and are obtained by treating the equation in vorticity form and reducing the latter to a random nonlinear parabolic equation. The solution has maximal regularity in the spatial variables and is weakly continuous in $(L^3 \cap L^{\frac{3p}{4p-6}})^3$ with respect to the time variable. Furthermore, we obtain the pathwise continuous dependence of solutions with respect to the initial data. In particular, one gets a locally unique solution of $3D$ stochastic Navier–Stokes equation in vorticity form up to some explosion stopping time τ adapted to the Brownian motion.

MSC: 60H15; 35Q30

Keywords: Stochastic Navier–Stokes equation; Vorticity; Biot–Savart operator

1. Introduction

Consider the stochastic $3D$ Navier–Stokes equation

* Corresponding author.
E-mail addresses: vb41@uaic.ro (V. Barbu), roeckner@math.uni-bielefeld.de (M. Röckner).

http://dx.doi.org/10.1016/j.jde.2017.06.020

Please cite this article in press as: V. Barbu, M. Röckner, Global solutions to random 3D vorticity equations for small initial data, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.06.020

$$\begin{aligned} dX-\Delta X\,dt+(X\cdot\nabla)X\,dt &= \sum_{i=1}^{N}(B_i(X)+\lambda_i X)d\beta_i(t)+\nabla\pi\,dt \\ &\qquad \text{on } (0,\infty)\times\mathbb{R}^3, \\ \nabla\cdot X=0 &\qquad \text{on } (0,\infty)\times\mathbb{R}^3, \\ X(0)=x &\qquad \text{in } (L^p(\mathbb{R}^3))^3, \end{aligned} \tag{1.1}$$

where $\lambda_i\in\mathbb{R}$, $x:\Omega\to\mathbb{R}^3$ is a random variable. Here π denotes the pressure and $\{\beta_i\}_{i=1}^N$ is a system of independent Brownian motions on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t\ge0}$, x is $\mathcal{F}_0$-measurable, and B_i are the convolution operators

$$B_i(X)(\xi)=\int_{\mathbb{R}^3}h_i(\xi-\bar\xi)X(\bar\xi)d\bar\xi=(h_i*X)(\xi),\ \xi\in\mathbb{R}^3, \tag{1.2}$$

where $h_i\in L^1(\mathbb{R}^3)$, $i=1,2,...,N$, and Δ is the Laplacian on $(L^2(\mathbb{R}^3))^3$.

It is not known whether (1.1) has a probabilistically strong solution in the mild sense for all time. Therefore, we shall rewrite (1.1) in vorticity form and then transform it into a random PDE, which we shall prove, has a global in time solution for $\mathbb{P}$-a.e. fixed $\omega\in\Omega$, provided the initial condition is small enough.

Consider the vorticity field

$$U=\nabla\times X=\text{curl }X \tag{1.3}$$

and apply the curl operator to equation (1.1). We obtain (see e.g. [4,9]) the transport vorticity equation

$$\begin{aligned} dU-\Delta U\,dt+((X\cdot\nabla)U-(U\cdot\nabla)X)dt &= \sum_{i=1}^{N}(h_i*U+\lambda_i U)d\beta_i \\ &\qquad \text{in } (0,\infty)\times\mathbb{R}^3, \\ U(0,\xi)=U_0(\xi)=(\text{curl}\,x)(\xi),\ \xi\in\mathbb{R}^3. \end{aligned} \tag{1.4}$$

The vorticity U is related to the velocity X by the equation

$$X(t,\xi)=K(U(t))(\xi),\ t\in(0,\infty),\ \xi\in\mathbb{R}^3, \tag{1.5}$$

where K is the Biot–Savart integral operator

$$K(u)(\xi)=-\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\xi-\bar\xi}{|\xi-\bar\xi|^3}\times u(\bar\xi)d\tilde\xi,\ \xi\in\mathbb{R}^3. \tag{1.6}$$

Then one can rewrite the vorticity equation (1.4) as

$$\begin{aligned} & dU-\Delta U\,dt+((K(U)\cdot\nabla)U-(U\cdot\nabla)K(U))dt \\ & \quad =\sum_{i=1}^{N}(h_i*U+\lambda_i U)d\beta_i\ \text{ in } (0,\infty)\times\mathbb{R}^3, \\ & U(0,\xi)=U_0(\xi),\ \xi\in\mathbb{R}^3. \end{aligned} \tag{1.7}$$

Please cite this article in press as: V. Barbu, M. Röckner, Global solutions to random 3D vorticity equations for small initial data, J. Differential Equations (2017), http://dx.doi.org/10.1016/j.jde.2017.06.020