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Asymptotic stabilization of inertial gradient dynamics with time-dependent viscosity ☆

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In homage to Felipe Alvarez, died prematurely

Abstract

In a Hilbert space $\mathcal{H}$, we study the asymptotic behavior, as time variable t goes to $+\infty$, of nonautonomous gradient-like inertial dynamics, with a time-dependent viscosity coefficient. Given $\Phi : \mathcal{H} \to \mathbb{R}$ a convex differentiable function, $\gamma(\cdot)$ a time-dependent positive damping term, we consider the second-order differential equation

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla\Phi(x(t)) = 0.$$

This system plays a central role in mechanics and physics in the asymptotic stabilization of nonlinear oscillators. Its importance in optimization was recently put to the fore by Su, Boyd, and Candès. They have shown that in the particular case $\gamma(t) = \frac{3}{t}$, this is a continuous version of the fast gradient method initiated by Nesterov, with $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}(\frac{1}{t^2})$ as $t \to +\infty$ in the worst case. Recently, in the case $\gamma(t) = \frac{\alpha}{t}$ with $\alpha > 3$, Attouch and Peypouquet have improved this result by showing the convergence of the trajectories to optimal solutions, and $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = o(\frac{1}{t^2})$ as $t \to +\infty$. For these questions, and the design of fast optimization methods, the tuning of the damping parameter $\gamma(t)$ is a subtle question, which we deal with in this paper in general. We obtain convergence rates for the values, and convergence results of the trajectories under general conditions on $\gamma(\cdot)$ which unify, and often improve the results already

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present in the literature. We complement these results by showing that they are robust with respect to perturbations.

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1. Introduction

$\mathcal{H}$ is a real Hilbert space, we write $\|x\|^2 = \langle x, x\rangle$ for $x \in \mathcal{H}$. For any differentiable function $F : \mathcal{H} \to \mathbb{R}$, its gradient is denoted by ∇F. Thus $F'(x)(y) = \langle \nabla F(x), y\rangle$. The first order (respectively second order) derivative at time t of a function $x(\cdot) : [t_0, +\infty[\to \mathcal{H}$ is denoted by $\dot{x}(t)$ (respectively $\ddot{x}(t)$).

1.1. Problem statement

Let $\Phi : \mathcal{H} \to \mathbb{R}$ be a convex function of class $\mathcal{C}^1$. We study the asymptotic behavior of the trajectories of the Inertial Gradient System

$$\ddot{x}(t) + \gamma(t)\dot{x}(t) + \nabla\Phi(x(t)) = 0, \quad t \geq t_0, \qquad (\mathrm{IGS}_\gamma)$$

where $\gamma : [t_0, +\infty[\to \mathbb{R}_+$ is a continuous damping function. This system plays a fundamental role in mechanics and physics. It models a nonlinear oscillator with viscous damping. Our motivation for studying this system with a general damping function $\gamma(t)$ comes mainly from optimization. Of course, the results we obtain in this paper also have a direct impact on the problems of physics and control theory.

The importance of taking a time-dependent damping coefficient in the evolution system (IGS_γ) is threefold. Our main concern in this paper is item *3*.

1. Mechanics and Physics: It describes the motion of a system, with mass equal to one, subject to a potential energy function Φ, and an isotropic linear damping with a viscosity parameter that may vanish asymptotically. This provides a simple model for a progressive reduction of the friction, possibly due to material fatigue.

2. Control theory: Another interpretation comes from the control theory, where $\gamma(\cdot)$ is a control variable that is designed to (asymptotically) effectively stabilize this nonlinear oscillator. Switching from the open-loop control system (IGS_γ) to a closed-loop system, where $\gamma(\cdot)$ is expressed as a feedback control, is a promising direction for future studies.

3. Fast numerical optimization methods: Taking a vanishing damping coefficient in system (IGS_γ) has recently been highlighted by Su, Boyd and Candès [32]. This is a key property for obtaining fast optimization methods. They have shown that, in the particular case $\gamma(t) = \frac{3}{t}$, (IGS_γ) is a continuous version of the fast gradient method initiated by Nesterov [24], with $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = \mathcal{O}(\frac{1}{t^2})$ in the worst case. Recently, Attouch and Peypouquet [6] and May [23] have improved this result by showing that $\Phi(x(t)) - \min_{\mathcal{H}} \Phi = o(\frac{1}{t^2})$ for $\gamma(t) = \frac{\alpha}{t}$ with $\alpha > 3$.

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